ON A METHOD OF DEALING WITH THE INTERSECTIONS OF PLANE CURVES*

BY

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Introduction.

1. The following is an investigation of certain theorems which I gave in the Proceedings of the London Mathematical Society, vol. 31 (1899), pp. 381–423, and which were later discussed in a simpler manner by DR. C. A. SCOTT in the Transactions of the American Mathematical Society, vol. 3 (1902), pp. 216–263. Initials, P and T, are used below for reference to these two sources respectively.

2. NOETHER'S fundamental theorem states that certain conditions are sufficient in order that a polynomial F in two variables x, y may be of the form $Af + B\phi$, where f, ϕ are given polynomials, and A, B undetermined polynomials. These conditions may be thus expressed: Change the origin to any point a, b, i. e., substitute $x + a, y + b$ for x, y in F, f, ϕ, and expand in ascending powers of x, y; then, whatever finite values a, b may have, F must be of the form $A'f + B'\phi$, where A', B' are undetermined infinite power series, which are entirely different for different origins. Any origin at a point of intersection of the curves f, ϕ supplies a certain number of conditional equations for the coefficients of F; the conditions which different origins supply are quite independent of one another, and there is no limit to the number of conditions which a single origin may supply. On the other hand, any origin which is not a point of intersection of f, ϕ supplies no conditions; for if f (say) does not vanish at the origin, F/f is a power series, so that F is of the form $A'f$, and therefore of the form $A'f + B'\phi$.

The two memoirs mentioned in § 1 relate to the conditions which are supplied for F in the most general case when the origin is a point at which the curves f, ϕ have multiple points of any order and complexity, and contact of any kind.

3. We modify the notation by writing S, C_1, C_2 for F, f, ϕ (C_1, C_2 being either given polynomials or power series), and P_1, P_2 for A', B'. We also write

$$C_1 = \Sigma a_q^p x^{p-q} y^q, \qquad C_2 = \Sigma b_q^p x^{p-q} y^q, \qquad S = \Sigma z_q^p x^{p-q} y^q \quad (p \geqq q \geqq 0).$$

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The subject of investigation is therefore the whole system of identical equations which must be satisfied by the coefficients z_q^p of S in order that the identity $S = C_1P_1 + C_2P_2$ may exist, where P_1, P_2 are undetermined power series.

It should be observed first of all that if the equation $\Sigma\alpha_q^p z_q^p = 0$ is identically satisfied, then all the equations $\Sigma\alpha_q^p z_{q-m}^{p-l} = 0$ $(l \geqq m \geqq 0)$ are also satisfied; for

$$\Sigma z_{q-m}^{p-l} x^{p-q} y^q = x^{l-m} y^m S = C_1P_1 + C_2P_2.$$

It is easily proved conversely that the equation $\Sigma\alpha_q^p z_q^p = 0$, and consequently the one-set system consisting of the prime equation $\Sigma\alpha_q^p z_q^p = 0$ and all its derivates * $\Sigma\alpha_q^p z_{q-m}^{p-l} = 0$, is identically satisfied, provided the same one-set system is satisfied for C_1 and C_2. *Thus the whole system of identical equations satisfied by the z's consists of all the one-set systems which are satisfied by the coefficients of C_1 and of C_2; and we have simply to investigate the properties of the one-set systems of equations which are satisfied by both C_1 and C_2.*

4. The work given below covers practically the same ground as Miss SCOTT's memoir T; and aims at dealing with the subject in an exact and general manner, although in many places the argument is based on particular examples. I have made several attempts at different times to place the theory on a purely algebraic foundation (cf. P § 8); but should probably not have succeeded except for the help derived from T. I have followed T in regarding the one-set theorem as the fundamental one, from which the whole subject is most naturally developed, and have found the diagrams of T § 32 of the greatest assistance in proving more than one important property (§§ 15–19). The principal theorems proved are:

(i) *The one-set theorem* (§§ 7–13). All the one-sets which two given curves, or series of coefficients, satisfy make up a single one-set; provided the two curves have not a common branch through the origin (§ 20).

Conversely, the complete solution of any one-set system of equations is expressible by means of two particular solutions (§ 14). This converse is not proved in T, and is not a consequence of the theorem itself.

(ii) *The t-set theorem* (§§ 22, 23). All the one-sets which $t + 1$ independent curves, or series, satisfy make up t independent one-sets, i. e., a t-set; provided the $t + 1$ curves have not all a common branch through the origin. This may be said to include its own converse.

(iii) The number of (ordinary) points to which the whole intersection of any two given curves at the origin is equivalent is equal to the number of independent equations in the one-set determined by the two curves (§ 18).

In the proof of (i) in T §§ 14–24, so many different cases are encountered as

* $\Sigma\alpha_q^p z_q^{p-1}$, $\Sigma\alpha_q^p z_q^{p-2}$, $\cdots$ are called the 1st, 2d, $\cdots$ x-derivates of the expression $\Sigma\alpha_q^p z_q^p$, and $\Sigma\alpha_q^p z_{q-1}^{p-1}$, $\Sigma\alpha_q^p z_{q-2}^{p-2}$, $\cdots$, the 1st, 2d, $\cdots$ y-derivates. Also E (or E_0^0) and E_m^l are written respectively for $\Sigma\alpha_q^p z_q^p$ and $\Sigma\alpha_q^p z_{q-m}^{p-l}$.

to create a doubt whether some possibilities may not have been overlooked, such as, for example, the possibility of A_0 vanishing in T § 20. I believe, however, that no important point is omitted; although I am not satisfied that all the points noted are sufficiently proved. In regard to (ii), I think the proof given in T § 25 is quite incorrect, and that (ii) cannot be proved apart from difficult considerations which are not involved in (i). An error also occurs in the proof of (iii) in T § 27, which invalidates the proof of T § 29.

5. The proof of (ii) in T § 25 is based on the assumption that if $m+1$ one-sets of degree p are satisfied by $t+1$ given independent curves, then the process of ascent *for two of the curves* will yield m equations of degree $p+1$ (the memoir says p which I take to be a misprint for $p+1$). But the process of ascent, as I understand it, is a general process, applicable to all curves which satisfy the $m+1$ one-sets of degree p; and it is only the result that is applied to particular curves. This general process, when applied to $m+1$ one-sets of degree p, does not in general give m equations of degree $p+1$. What must be shown is that if all the one-sets of degree $\leqq p$ satisfied by the curves make up an $(m+1)$-set, where $m+1>t$, then the process of ascent yields one or more equations of degree $p+1$, containing sufficient undetermined constants to allow of their being satisfied by the $t+1$ curves. If the above criticism is due to a misapprehension of the meaning of T § 25, it may still be maintained that the reasoning falls far short of a complete proof of theorem (ii).

6. The statement of T § 27 is: "If an expression E and its first k y-derivates $E_1^1, E_2^2, \cdots, E_k^k$, as well as all the x-derivates of these, vanish for two curves that have at the origin a multiple point of order k, then all the derivates of E vanish, so that E is a member of the one-set system proper to the point." If true, this would materially simplify the proof of (iii) in P §§ 12–17; but it is assumed that no equations which involve the coefficients of powers of y only can be satisfied for both curves, when the axes have no special relation to the curves. This is clearly a mistake, when, as is the case in T § 27, the equations do not belong to a one-set system.

Take, as an example, the point whose prime equation is

$$z_0^3 + mz_1^3 + m^2z_2^3 + m^3z_3^3 = 0,$$

the derivates of which are

$$z_0^2 + mz_1^2 + m^2z_2^2 = z_0^1 + mz_1^1 = z_0^0 = 0.$$

The two curves C_1, C_2 which determine this point have only a linear branch through the origin, so that $k=1$. Now take

$$E = z_0^4 + mz_1^4 + m^2z_2^4 + m^3z_3^4 + m^4z_4^4 + \alpha z_4^4 + \beta z_3^3 + \gamma z_2^2 + \delta z_1^1;$$

then

$$E_1^1 = mz_0^3 + m^2z_1^3 + m^3z_2^3 + m^4z_3^3 + \alpha z_3^3 + \beta z_2^2 + \gamma z_1^1.$$

All the x-derivates of E and E_1^1 vanish for C_1 and C_2, and $\alpha, \beta, \gamma, \delta$ can be so chosen that E and E_1^1 also vanish for C_1 and C_2; yet E is not a member of the one-set system proper to the point. It may be noticed that

$$E_1^1 - mE_0^1 = \alpha z_3^3 + \beta z_2^2 + \gamma z_1^1;$$

but this expression can, and does, vanish for both curves, notwithstanding the fact that it involves coefficients of powers of y only.

The one-set theorem.

7. Let $C_1 = \Sigma a_q^p x^{p-q} y^q$, $C_2 = \Sigma b_q^p x^{p-q} y^q$, be two given polynomials or power series in which no terms are present of degree less than i. We shall be concerned chiefly with the series of elements formed by their coefficients, viz.:

$$a_0^i, a_1^i, \cdots, a_i^i, a_0^{i+1}, a_1^{i+1}, \cdots, a_{i+1}^{i+1}, a_0^{i+2}, \cdots,$$
$$b_0^i, b_1^i, \cdots, b_i^i, b_0^{i+1}, b_1^{i+1}, \cdots, b_{i+1}^{i+1}, b_0^{i+2}, \cdots.$$

In each series the first $\frac{1}{2}i(i+1)$ elements vanish, and also all elements outside a certain range if C_1, C_2 are polynomials. All these vanishing elements should, however, be considered as belonging to the series. We may assume that no other elements vanish.* It is only essential, however, that *we assume a_i^i not to vanish.*

We enquire first, what are the one-sets which both series satisfy? In order that $\Sigma\zeta_q^p z_q^p = 0$ may be one it is necessary and sufficient (§ 3) that the equation $\Sigma\zeta_q^p z_q^p = 0$, and all its derivates, should be satisfied both when $z_q^p = a_q^p$, and when $z_q^p = b_q^p$. This gives at once the equations for the ζ's; and the general solution (involving arbitrary linear parameters), being substituted in $\Sigma\zeta_q^p z_q^p = 0$, will give all the equations (both prime and derivate) of the required one-set systems, by equating to zero the coefficients of the arbitrary parameters involved in the ζ's. Moreover any particular solution of the ζ's supplies the coefficients of a one-set $\Sigma\zeta_p^p z_q^p = 0$, which may however be a derivate of a more extensive one-set.

8. The ζ equations are

$$\Sigma\zeta_q^p a_q^p = 0, \quad \Sigma\zeta_q^p a_{q-1}^{p-1} = 0, \quad \Sigma\zeta_q^p a_{q-1}^{p-1} = 0, \quad \Sigma\zeta_q^p a_q^{p-2} = 0, \quad \cdots,$$

and

$$\Sigma\zeta_q^p b_q^p = 0, \quad \Sigma\zeta_q^p b_q^{p-1} = 0, \quad \Sigma\zeta_q^p b_{q-1}^{p-1} = 0, \quad \Sigma\zeta_q^p b_q^{p-2} = 0, \quad \cdots,$$

which may also be written

$$\Sigma a_q^p \zeta_q^p = 0, \quad \Sigma a_q^p \zeta_q^{p+1} = 0, \quad \Sigma a_q^p \zeta_{q+1}^{p+1} = 0, \quad \Sigma a_q^p \zeta_q^{p+2} = 0, \quad \cdots,$$
$$\Sigma b_q^p \zeta_q^p = 0, \quad \Sigma b_q^p \zeta_q^{p+1} = 0, \quad \Sigma b_q^p \zeta_{q+1}^{p+1} = 0, \quad \Sigma b_q^p \zeta_q^{p+2} = 0, \quad \cdots.$$

* If C_1, C_2 are transformed to C_1', C_2' when $x = lx' + my'$, $y = l'x' + m'y'$ $(l/m \neq l'/m')$, it can be easily proved that a one-to-one correspondence exists between the one-sets satisfied by C_1, C_2 and the one-sets satisfied by C_1', C_2'. The same is true for the transformation $x = lx' + my' +$ higher terms, $y = l'x' + m'y' +$ higher terms.

These are best exhibited by simply writing down the array of the coefficients:

$$
\begin{array}{cccccccccccccc}
\zeta_0^i & \zeta_1^i & \zeta_2^i & \cdots & \zeta_i^i & \zeta_0^{i+1} & \zeta_1^{i+1} & \zeta_2^{i+1} & \cdots & \zeta_{i+1}^{i+1} & \zeta_0^{i+2} & \zeta_1^{i+2} & \zeta_2^{i+2} & \cdots & \zeta_{i+2}^{i+2} & \cdots \\
\hline
a_0^i & a_1^i & a_2^i & \cdots & a_i^i & a_0^{i+1} & a_1^{i+1} & a_2^{i+1} & \cdots & a_{i+1}^{i+1} & a_0^{i+2} & a_1^{i+2} & a_2^{i+2} & \cdots & a_{i+2}^{i+2} & \cdots \\
b_0^i & b_1^i & b_2^i & \cdots & b_i^i & b_0^{i+1} & b_1^{i+1} & b_2^{i+1} & \cdots & b_{i+1}^{i+1} & b_0^{i+2} & b_1^{i+2} & b_2^{i+2} & \cdots & b_{i+2}^{i+2} & \cdots \\
 & & & & & a_0^i & a_1^i & a_2^i & \cdots & 0 & a_0^{i+1} & a_1^{i+1} & a_2^{i+1} & \cdots & 0 & \cdots \\
 & & & & & 0 & a_0^i & a_1^i & \cdots & a_i^i & 0 & a_0^{i+1} & a_1^{i+1} & \cdots & a_{i+1}^{i+1} & \cdots \\
 & & & & & b_0^i & b_1^i & b_2^i & \cdots & 0 & b_0^{i+1} & b_1^{i+1} & b_2^{i+1} & \cdots & 0 & \cdots \\
 & & & & & 0 & b_0^i & b_1^i & \cdots & b_i^i & 0 & b_0^{i+1} & b_1^{i+1} & \cdots & b_{i+1}^{i+1} & \cdots \\
 & & & & & & & & & & a_0^i & a_1^i & a_2^i & \cdots & 0 & \cdots \\
 & & & & & & & & & & 0 & a_0^i & a_1^i & \cdots & 0 & \cdots \\
 & & & & & & & & & & 0 & 0 & a_0^i & \cdots & a_i^i & \cdots \\
 & & & & & & & & & & b_0^i & b_1^i & b_2^i & \cdots & 0 & \cdots \\
 & & & & & & & & & & 0 & b_0^i & b_1^i & \cdots & 0 & \cdots \\
 & & & & & & & & & & 0 & 0 & b_0^i & \cdots & b_i^i & \cdots \\
 & & & & & & & & & & & & & & & \cdots \\
 & & & & & & & & & & & & & & & \cdots
\end{array}
$$

Thus the equations divide themselves into sets, according to the index of ζ in the leading terms. Corresponding to these we have sets of elements, and leading elements of a set, viz. the array of coefficients of ζ's with lowest index in the set. This index is called the degree of the set and of the corresponding set of equations. As they stand, the sets of elements contain two, four, six, etc., rows respectively; but we may have to modify and diminish the sets, while keeping the whole array or system of equations complete, in order that, in the final form, the rows of leading elements in each set may be linearly independent. We will suppose this done for the present, and proceed to mention the chief properties of the equations, leaving the proofs of (ii), (iv), and (vi) to be given later.

(i) Each ζ with an index less than i is arbitary, since its coefficient is zero in all the equations.

Each ζ with an index greater than the limit index is zero, as explained in (iii).

(ii) Each set in the modified array contains either one or two more rows (or equations) than the preceding set (§ 9). If only one more, the set is said to be *irregular*; if two more, the set is *regular*.

The first set will be irregular if C_1, C_2 have precisely the same tangents at the origin. If the first $j - i$ sets are irregular, an integral polynomial S exists such that $C_2 + C_1 S$ has a multiple point of order j at the origin. Also, as C_2 may be replaced by $C_2 + C_1 S$ for the purpose of finding the one-sets of C_1, C_2, this case corresponds to two curves having multiple points of different orders i, j at the origin.

(iii) There cannot be more than i regular sets altogether; and at the ith regular set the ζ's vanish. For the number of ζ's in the leading terms of the ith regular set of equations is equal to the number of equations in that set; and the same is true for every later set, by (ii). Hence all ζ's with an index greater than a certain limit index (viz., the index of the ζ's in the leading terms of the set preceding the ith regular set) are expressible in terms of ζ's whose indices exceed any assigned index, however great, and so cannot be retained. Thus all ζ's which occur in the ith regular set must be taken to be zero.

This may also be proved in another way. Let n be the index of the ζ's in the leading terms of the ith regular set, so that $n+1$ is the number of rows of the set. Now the value of z_q^p (the coefficient of $x^{p-q}y^q$ in $C_1P_1 + C_2P_2$, P_1, P_2 being arbitrary) is obtained by taking the sum of the products of a column of arbitrary elements (or parameters) with the column of elements corresponding to ζ_q^p. Hence, in $z_0^n, z_1^n, \cdots, z_n^n$, $n+1$ significant parameters are involved, viz., the elements of the arbitrary column corresponding to the $n+1$ rows of the ith regular set, which do not appear at all in the z's with index $< n$. Thus $z_0^n, z_1^n, \cdots, z_n^n$ cannot appear permanently in any identical equation (after their coefficients have been collected), since the $n+1$ significant parameters cannot be eliminated; and, for the same reason, no z with an index $\geqq n$ can appear. But $\Sigma\zeta_q^p z_q^p = 0$ is an identical equation; hence ζ_q^p vanishes when $p \geqq n$.

(iv) If every set after the kth regular set is irregular, where $k < i$, then C_1, C_2 are divisible by a common power series K having a multiple point of order $i-k$ at the origin (§ 20). In this case the limit index mentioned in (iii) does not exist; and the whole intersection of C_1, C_2 at the origin is made up of K and the finite intersection of C_1/K and C_2/K at the origin.

(v) If $\zeta_q^p = \alpha_q^p$ is one solution of the ζ equations, it is easily seen that $\zeta_q^p = \alpha_q^{p+1}$, $\zeta_q^p = \alpha_{q-1}^{p+1}$, $\cdots \zeta_q^p = \alpha_{q-m}^{p+l}$ $(l \geqq m)$, $\cdots$ are other solutions (§ 8, line 6).

(vi) If $\zeta_q^p = \alpha_q^p$ is any particular solution such that the α's with the limit index do not vanish, then the several solutions $\zeta_q^p = \alpha_{q+s}^{p+r}$ $(r \geqq s)$ give the complete solution, i. e., the general solution is expressible by means of a single solution, viz., $\zeta_q^p = \Sigma\lambda_s^r \alpha_{q+s}^{p+r}$, where the λ's are arbitrary (§§ 10–13).

This last is the fundamental property of the ζ equations; and from it follows the one-set theorem § 4 (i). For all the one-sets which the series a, b satisfy are included in $\Sigma\zeta_q^p z_q^p = 0$, when ζ_q^p is given its general value; but

$$\Sigma\zeta_q^p z_q^p = \Sigma z_q^p \Sigma\lambda_s^r \alpha_{q+s}^{p+r} = \Sigma\lambda_s^r \Sigma\alpha_{q+s}^{p+r} z_q^p = \Sigma\lambda_s^r \Sigma\alpha_q^p z_{q-s}^{p-r};$$

so that the general equation $\Sigma\zeta_q^p z_q^p = 0$ gives the one-set $\Sigma\alpha_q^p z_q^p = 0$ and its derivates, and nothing more. This reasoning is not affected by the fact that the number of the λ's is not the least number of arbitrary parameters by means of which the general solution for ζ can be expressed.

9. We have now to examine the result of modifying the array of the coeffi-

cients (§ 8) so as to make the rows of leading elements in each set linearly independent. Suppose the first p sets are regular, and the $\overline{p+1}$th irregular. The leading elements of the pth and $\overline{p+1}$th sets are

$$\begin{array}{llll} a_0^i\ a_1^i\ a_2^i\ \cdots & & \text{and} & ;\ a_0^i\ a_1^i\ a_2^i\ \cdots \\ 0\ \ a_0^i\ a_1^i\ \cdots & & & 0\ \ a_0^i\ a_1^i\ \cdots \\ \quad p \text{ rows} & & & p+1 \text{ rows} \\ b_0^i\ b_1^i\ b_2^i\ \cdots & & & b_0^i\ b_1^i\ b_2^i\ \cdots \\ 0\ \ b_0^i\ b_1^i\ \cdots & & & 0\ \ b_0^i\ b_1^i\ \cdots \\ \quad p \text{ rows}, & & & p+1 \text{ rows}; \end{array}$$

and in the former the rows are linearly independent, but not in the latter. The latter array differs from the former only by the addition of the last a row, the last b row, and the last column. Hence the first p rows a and first p rows b in in the latter array are independent. These supply none but zero elements for the last column; hence the last a row, which has a non-vanishing element a_i^i in the last column, is an additional independent row. The last row of all must then be dependent on the first $2p+1$ rows, which are independent. We can therefore multiply the first $2p+1$ rows by such quantities that, when added to the last row, the leading elements in that row all vanish. The row of the whole array becomes then a row belonging to the $\overline{p+2}$th set, with leading terms ζ^{i+p+1}, and is

$$c_0^{i+p+1},\ c_1^{i+p+1},\ \cdots,\ c_{i+p+1}^{i+p+1},\ c_0^{i+p+2},\ \cdots,\ c_{i+p+2}^{i+p+2},\ \cdots.$$

The leading elements of the $\overline{p+2}$th set consist now of $p+2$ rows $a_0^i, \cdots, a_i^i$, $p+2$ rows $b_0^i, \cdots, b_i^i$, and 1 row $c_0^{i+p+1}, \cdots, c_{i+p+1}^{i+p+1}$. By treating the first $p+1$ rows a and first $p+1$ rows b in the same way as in the $\overline{p+1}$th set, the $\overline{p+1}$th row b is replaced by a row c in the $\overline{p+3}$th set; similarly by means of the $p+1$ rows a after the first, and the $p+1$ rows b after the first, the $\overline{p+2}$th row b is replaced by a second row c in the $\overline{p+3}$th set. The two rows c in the $\overline{p+3}$th set are

$$\begin{array}{llll} c_0^{i+p+1} & c_1^{i+p+1} \cdots c_{i+p+1}^{i+p+1} & 0 & c_0^{i-p+2}\ \ c_1^{i+p+2} \cdots \\ 0 & c_0^{i+p+1} \cdots c_{i+p}^{i+p+1} & c_{i+p+1}^{i+p+1} & 0\ \ c_0^{i+p+2} \cdots. \end{array}$$

Thus the leading elements in the $\overline{p+2}$th set reduce to $p+2$ rows a, p rows b, and 1 row c. Those of the $\overline{p+3}$th set similarly reduce to $p+3$ rows a, p rows b, and 2 rows c; and so on for the remaining sets, the number of a rows and c rows increasing by 1 in each set, and the number of b rows remaining stationary and equal to p. Thus the effect of an irregularity in the $\overline{p+1}$th set

is to reduce the whole number of rows in each set from the $\overline{p+1}$th onwards by 1, the qth set ($q>p$) having its q rows b altered to p rows b and $q-p-1$ rows c.

We may now consider the effect of a second irregularity, which may occur at the $\overline{p+2}$th or any later set. Suppose it occurs at the $\overline{q+2}$th set ($q \geqq p$). As before, the last row of leading elements of the $\overline{q+2}$th set, i. e., the $\overline{q-p+1}$th row c, must be dependent on the rest, and the row of the whole array is replaced by a row d belonging to the $\overline{q+3}$th set. The same reasoning applies as before, the only difficulty occurring with the b rows, the number of which remains stationary after the pth set. We may, however, reinsert the b rows which have disappeared for the purpose of deducing the d rows. The result is that the $\overline{q+3}$th set will have $q+3$ rows a, p rows b, $q-p$ rows c, and 1 row d; and in succeeding sets the number of a and d rows will increase by one in each set, and the number of b and c rows will remain stationary, while the whole number of rows in each set from the $\overline{q+2}$th onwards will be again diminished by 1. This process of modification must be continued until we arrive at the ith regular set with leading elements linearly independent. From and after this point the ζ's vanish (§ 8, iii). The system of ζ equations may then be said to be prepared for solution. The solution is obtained by starting with the set preceding the ith regular set, and working backwards through the sets.

10. We have already seen in § 8 (v) that a single solution $\zeta_q^p = \alpha_q^p$ gives other solutions, viz., the $(s+1)$th solution of the $(r+1)$th set of solutions is $\zeta_q^p = \alpha_{q+s}^{p+r}$ ($r \geqq s$). The array of these solutions, supposing l the limit index, is

$$
\begin{array}{c|ccccccccccccc}
 & \zeta_0^0 & \zeta_0^1 & \zeta_1^1 & \zeta_0^2 & \cdots & \zeta_0^{l-2} & \cdots & \zeta_{l-2}^{l-2} & \zeta_0^{l-1} & \cdots & \zeta_{l-1}^{l-1} & \zeta_0^l & \cdots & \zeta_l^l \\
\hline
E_0^0 & \alpha_0^0 & \alpha_0^1 & \alpha_1^1 & \alpha_0^2 & \cdots & \alpha_0^{l-2} & \cdots & \alpha_{l-2}^{l-2} & \alpha_0^{l-1} & \cdots & \alpha_{l-1}^{l-1} & \alpha_0^l & \cdots & \alpha_l^l \\
E_0^1 & \alpha_0^1 & \alpha_0^2 & \alpha_1^2 & \alpha_0^3 & \cdots & \alpha_0^{l-1} & \cdots & \alpha_{l-2}^{l-1} & \alpha_0^l & \cdots & \alpha_{l-1}^l & & & \\
E_1^1 & \alpha_1^1 & \alpha_1^2 & \alpha_2^2 & \alpha_1^3 & \cdots & \alpha_1^{l-1} & \cdots & \alpha_{l-1}^{l-1} & \alpha_1^l & \cdots & \alpha_l^l & & & \\
E_0^2 & \alpha_0^2 & \alpha_0^3 & \alpha_1^3 & \alpha_0^4 & \cdots & \alpha_0^l & \cdots & \alpha_{l-2}^l & & & & & & \\
E_1^2 & \alpha_1^2 & \alpha_1^3 & \alpha_2^3 & \alpha_1^4 & \cdots & \alpha_1^l & \cdots & \alpha_{l-1}^l & & & & & & \\
E_2^2 & \alpha_2^2 & \alpha_2^3 & \alpha_3^3 & \alpha_2^4 & \cdots & \alpha_2^l & \cdots & \alpha_l^l & & & & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & & & \\
\hline
 & z_0^0 & z_0^1 & z_1^1 & z_0^2 & \cdots & z_0^{l-2} & \cdots & z_{l-2}^{l-2} & z_0^{l-1} & \cdots & z_{l-1}^{l-1} & z_0^l & \cdots & z_l^l
\end{array}
$$

This is also the array of the coefficients of the one-set system of equations consisting of $E_0^0 = \Sigma \alpha_q^p z_q^p = 0$ and all its derivates $E_s^r = 0$, which we will call the z equations. The z equations divide into sets, according to the degree, or index of z in the leading terms, giving this name to the terms with highest index in the z equations, and to the terms with lowest index in the ζ equations. We modify the z array in the same way as the ζ array, so that the rows of leading elements in each set may be linearly independent. The reasoning of § 9 shows that, after modifica-

tion, the number of rows in any set is at most 1 more than in the previous set; if it is 1 more, the set is called regular, otherwise it is irregular. *An irregular set has the same number of rows as the preceding set.* For, in the set of equations of degree n, we cannot eliminate all the leading terms $z_0^n, z_1^n, \cdots z_n^n$, since the rows of leading elements are independent. Nor can we eliminate $z_0^n, z_1^n, \cdots, z_{n-1}^n$, as that would make z_n^n a linear function of z's with lower index, and z_i^i also, by taking the $(n-i)$th y-derivate; so that $z_i^i = 0$, which is contrary to the hypothesis $a_i^i \neq 0$.* We here use the fact that every z with index $< i$ vanishes, not assuming, however, that it is a consequence of the one-set $\Sigma \alpha_q^p z_q^p = 0$. Now the coefficients of $z_0^n, z_1^n, \cdots, z_{n-1}^n$ are leading elements in the set of degree $n-1$. Hence the set of degree $n-1$ has as many rows, and if irregular has the same number of rows, as the set of degree n.

11. Two *corresponding* arrays are defined as those which have the same number of columns and are such that the sum of the products of the elements of any row in one and any row in the other is zero. When each array consists of independent rows we shall call them corresponding independent arrays. It is clear that if the whole number of rows of two corresponding independent arrays is equal to the number of columns, each array represents the complete solution of the equations given by the other array; from which it follows also that the whole number of rows in the two arrays cannot exceed the number of columns in either array.

The modified ζ and z arrays are corresponding independent arrays, and so also are the arrays of leading elements of any two sets of the same degree. Hence, to prove that the z array gives the complete solution of the ζ equations, it has only to be shown that the leading elements of any two sets of the same degree together form a square array; for then the whole ζ and z arrays also form a square array. This amounts to proving that if the set of degree n in the ζ array is regular, then the set of degree $n-1$ in the z array is regular. We shall prove this for a particular example, sufficiently complicated to make the general reasoning clear.

12. Suppose that $i = 5$, and that the sets of degree 5, 6, 9, 11, 12 of the ζ array are regular; then we have to prove that the sets of degree 4, 5, 8, 10, 11 of the z array are regular. We know that the last is regular, since by hypothesis, the α's with limit index 11 do not vanish. We shall assume that the sets of degree 10 and 8 are regular also, in which case the sets of degree 9, 7, 6 must be irregular, and prove that the sets of degree 5 and 4 are regular; this will include all the steps of the proof for the sets of degree 10 and 8. The ζ and z arrays are exhibited by the following table:

* This proof is taken from T § 7; but is simplified by reason of the assumption $a_i^i \neq 0$.

ζ_0^4	ζ_0^5	ζ_0^6	ζ_0^7	ζ_0^8	ζ_0^9	ζ_0^{10}	ζ_0^{11}	ζ_0^{12}			
	a_0^5	a_0^6	a_0^7	a_0^8	a_0^9	a_0^{10}	a_0^{11}	a_0^{12}	1	row	reg.
	b_0^5	b_0^6	b_0^7	b_0^8	b_0^9	b_0^{10}	b_0^{11}	b_0^{12}	1	"	
		a_0^5	a_0^6	a_0^7	a_0^8	a_0^9	a_0^{10}	a_0^{11}	2	rows	reg.
		b_0^5	b_0^6	b_0^7	b_0^8	b_0^9	b_0^{10}	b_0^{11}	2	"	
			a_0^5	a_0^6	a_0^7	a_0^8	a_0^9	a_0^{10}	3	"	irr.
			b_0^5	b_0^6	b_0^7	b_0^8	b_0^9	b_0^{10}	2	"	
				a_0^5	a_0^6	a_0^7	a_0^8	a_0^9	4	"	irr.
				b_0^5	b_0^6	b_0^7	b_0^8	b_0^9	2	"	
					a_0^5	a_0^6	a_0^7	a_0^8	5	"	reg.
					b_0^5	b_0^6	b_0^7	b_0^8	2	"	
					c_0^9	c_0^{10}	c_0^{11}	c_0^{12}	1	"	
						a_0^5	a_0^6	a_0^7	6	"	irr.
						b_0^5	b_0^6	b_0^7	2	"	
						c_0^9	c_0^{10}	c_0^{11}	1	"	
							a_0^5	a_0^6	7	"	reg.
							b_0^5	b_0^6	2	"	
							c_0^9	c_0^{10}	1	"	
							d_0^{11}	d_0^{12}	1	"	
								a_0^5	8	"	reg.
								b_0^5	2	"	
								c_0^9	1	"	
								d_0^{11}	2	"	
α_0^4	α_0^5	α_0^6	α_0^7	α_0^8	α_0^9	α_0^{10}	α_0^{11}		1	row	reg.
α_0^5	α_0^6	α_0^7	α_0^8	α_0^9	α_0^{10}	α_0^{11}			2	rows	reg.
α_0^6	α_0^7	α_0^8	α_0^9	α_0^{10}	α_0^{11}				2	"	irr.
α_0^7	α_0^8	α_0^9	α_0^{10}	α_0^{11}					2	"	reg.
β_0^4	β_0^5	β_0^6	β_0^7	β_0^8					1	row	
α_0^8	α_0^9	α_0^{10}	α_0^{11}						2	rows	irr.
β_0^5	β_0^6	β_0^7	β_0^8						1	row	
α_0^9	α_0^{10}	α_0^{11}							2	rows	irr.
β_0^6	β_0^7	β_0^8							1	row	
α_0^{10}	α_0^{11}								2	rows	reg.
β_0^7	β_0^8								1	row	
γ_0^4	γ_0^5								1	"	
α_0^{11}									2	rows	reg.
β_0^8									1	row	
γ_0^5									2	rows	
z_0^4	z_0^5	z_0^6	z_0^7	z_0^8	z_0^9	z_0^{10}	z_0^{11}	z_0^{12}			

Only the merest outline of the arrays is given; but the filling up will be

understood from what has been already said. Thus the leading elements of the third set of the upper array and fourth set of the lower array are respectively

$$\begin{array}{cccccccc} \boldsymbol{a_0^5} & a_1^5 & a_2^5 & a_3^5 & a_4^5 & a_5^5 & 0 & 0 \\ 0 & a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 & a_5^5 & 0 \\ 0 & 0 & a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 & a_5^5 \\ \boldsymbol{b_0^5} & b_1^5 & b_2^5 & b_3^5 & b_4^5 & b_5^5 & 0 & 0 \\ 0 & b_0^5 & b_1^5 & b_2^5 & b_3^5 & b_4^5 & b_5^5 & 0 \end{array} \quad \text{and} \quad \begin{array}{ccccccccc} \alpha_0^{11} & \alpha_1^{11} & \alpha_2^{11} & \alpha_3^{11} & \alpha_4^{11} & \alpha_5^{11} & \alpha_6^{11} & \alpha_7^{11} & \alpha_8^{11} \\ \alpha_1^{11} & \alpha_2^{11} & \alpha_3^{11} & \alpha_4^{11} & \alpha_5^{11} & \alpha_6^{11} & \alpha_7^{11} & \alpha_8^{11} & \alpha_9^{11} \\ \beta_0^8 & \beta_1^8 & \beta_2^8 & \beta_3^8 & \beta_4^8 & \beta_5^8 & \beta_6^8 & \beta_7^8 & \beta_8^8 \end{array}$$

A curious property of the z array, which we do not stay to prove, is that the determinants formed by consecutive columns of the leading elements of any set which is one degree higher than an irregular set are in geometrical progression. The connected property of the ζ array is not expressed so simply.

13. We have to prove that the set of degree 5 in the z array is regular, i. e., has 4 independent rows. If it is not regular, then, by joining it on to the two preceding irregular sets, we see that the following array must vanish:

$$\begin{array}{cccccccccc} \alpha_0^9 & \cdots & \alpha_5^9 & \alpha_0^{10} & \cdots & \alpha_6^{10} & \alpha_0^{11} & \cdots & \alpha_7^{11} \\ \alpha_1^9 & \cdots & \alpha_6^9 & \alpha_1^{10} & \cdots & \alpha_7^{10} & \alpha_1^{11} & \cdots & \alpha_8^{11} \\ \beta_0^6 & \cdots & \beta_5^6 & \beta_0^7 & \cdots & \beta_6^7 & \beta_0^8 & \cdots & \beta_7^8 \\ \beta_1^6 & \cdots & \beta_6^6 & \beta_1^7 & \cdots & \beta_7^7 & \beta_1^8 & \cdots & \beta_8^8 \\ \alpha_0^{10} & \cdots & \alpha_5^{10} & \alpha_0^{11} & \cdots & \alpha_6^{11} & & & \\ \alpha_1^{10} & \cdots & \alpha_6^{10} & \alpha_1^{11} & \cdots & \alpha_7^{11} & & & \\ \beta_0^7 & \cdots & \beta_5^7 & \beta_0^8 & \cdots & \beta_6^8 & & & \\ \alpha_0^{11} & \cdots & \alpha_5^{11} & & & & & & \\ \alpha_1^{11} & \cdots & \alpha_6^{11} & & & & & & \\ \beta_0^8 & \cdots & \beta_5^8 & & & & & & \end{array}$$

The 4th row here is reinserted, since it belongs to the complete z array, and gives rise to the row γ in the set of degree 5. The other 9 rows are independent, as already proved, so that the 4th row is dependent on the other 9, on the supposition that the array vanishes. We may then add further columns to the array, so that the new array also vanishes, by properly choosing the values of the elements added in the 4th row. We shall add three such columns, containing three new elements β_7^6, β_8^7, β_9^8 in the 4th row. Then by comparing the sets of degree 6, 7, 8 in the ζ and z arrays the following are seen to be corresponding arrays, the 4th row of the lower array being added, since it is dependent on the others:

a_0^5	...	a_0^6	...	a_0^7	...	2 rows						
b_0^5	...	b_0^6	...	b_0^7	...	2 "						
		a_0^5	...	a_0^6	...	3 "						
		b_0^5	...	b_0^6	...	2 "						
				a_0^5	...	4 "						
				b_0^5	...	2 "						
α_0^9	...	α_6^9	α_0^{10}	...	α_7^{10}	α_0^{11}	...	α_8^{11}				
α_1^9	...	α_7^9	α_1^{10}	...	α_8^{10}	α_1^{11}	...	α_9^{11}				
β_0^5	...	β_5^5	β_0^7	...	β_7^7	β_0^8	...	β_8^8				
β_1^5	...	β_7^5	β_1^7	...	β_8^7	β_1^8	...	β_9^8				
α_0^{10}	...	α_6^{10}	α_0^{11}	...	α_7^{11}							
α_1^{10}	...	α_7^{10}	α_1^{11}	...	α_8^{11}							
β_0^7	...	β_6^7	β_0^8	...	β_7^8							
α_0^{11}	...	α_6^{11}										
α_1^{11}	...	α_7^{11}										
β_0^8	...	β_6^8										

Hence the following arrays also correspond :

a_0^5	...	a_0^6	...	a_0^7	...	3 rows						
b_0^5	...	b_0^6	...	b_0^7	...	3 "						
		a_0^5	...	a_0^6	...	4 "						
		b_0^5	...	b_0^6	...	2 "						
				a_0^5	...	5 "						
				b_0^5	...	2 "						
α_0^9	...	α_7^9	α_0^{10}	...	α_8^{10}	α_0^{11}	...	α_9^{11}				
α_1^9	...	α_8^9	α_1^{10}	...	α_9^{10}	α_1^{11}	...	α_{10}^{11}				
β_0^6	...	β_7^6	β_0^7	...	β_8^7	β_0^8	...	β_9^8				
α_0^{10}	...	α_7^{10}	α_0^{11}	...	α_8^{11}							
α_1^{10}	...	α_8^{10}	α_1^{11}	...	α_9^{11}							
β_0^7	...	β_7^7	β_0^8	...	β_8^8							
α_0^{11}	...	α_7^{11}										
α_1^{11}	...	α_8^{11}										
β_0^8	...	β_7^8										

Here the only row not given by the sets of degree 7, 8, 9 of the table is the third row of the lower array, which takes the place of the 3d and 4th rows of the lower array in the previous step. Both arrays consist of independent rows, the last row of the 1st set in the upper array being equivalent to a row c in the 3d set. But the number of rows in the two arrays combined is greater than the number of columns, which is impossible. The set of degree 5 in the z array must therefore have 4 independent rows, and is regular.

Again, the set of degree 4 of the z array is regular; for the first 4 rows are independent (§ 10), and, if the 5th row is dependent on them, we can add an element γ_6^5 such that the array

$$\begin{array}{l} \alpha_0^{11} \cdots \alpha_5^{11} \\ \alpha_1^{11} \cdots \alpha_6^{11} \\ \beta_0^8 \cdots \beta_5^8 \\ \gamma_0^5 \cdots \gamma_5^5 \\ \gamma_1^5 \cdots \gamma_6^5 \end{array}$$

vanishes. To this corresponds the array

$$\begin{array}{l} a_0^5 \cdots a_5^5 \\ b_0^5 \cdots b_5^5; \end{array}$$

hence the arrays

$$\begin{array}{ll} a_0^5 \cdots & 2 \text{ rows} \\ b_0^5 \cdots & 2 \text{ rows} \\ \hline \alpha_0^{11} \cdots \alpha_6^{11} & \\ \alpha_1^{11} \cdots \alpha_7^{11} & \\ \beta_0^8 \cdots \beta_6^8 & \\ \gamma_0^5 \cdots \gamma_6^5 & \end{array}$$

correspond; which is impossible, for the same reason as before.

The above reasoning, although applied to a particular example, is perfectly general, and shows that the general solution of any system of ζ equations, which includes i regular sets of equations, is $\zeta_q^p = \Sigma\lambda_s^r \alpha_{q+s}^{p+r}$, where $\zeta_q^p = \alpha_q^p$ is a particular solution. From this the one-set theorem follows, as in § 8 (vi).

14. The converse theorem that the complete solution of any one-set system of equations is expressible by means of two particular solutions a, b (viz. $z_q^p = \Sigma\lambda_s^r a_{q-s}^{p-r} + \Sigma\mu_s^r b_{q-s}^{p-r}$) is proved in a similar way. Taking the

example already considered, we suppose the z array given, and have to prove that there are two particular solutions a, b of the z equations such that, if the ζ array be formed for the series a, b, the set of degree n in the ζ array is regular if the set of degree $n-1$ in the z array is regular. We assume that the z equations do not require z_5^5 to vanish (footnote § 7). The series a is then any solution in which $a_5^5 \neq 0$, and the series b is any other solution which supplies an additional independent row in the earliest set of the ζ array where it happens to be needed. In the example considered this is the first set, so that a suitable choice for the solution b would be any one in which $b_5^5 = 0$, provided $b_0^5, b_1^5, b_2^5, b_3^5, b_4^5$ do not all vanish.

If the set of degree 6 in the ζ array were not then regular, γ_6^5 could be so chosen that the two following arrays correspond:

$$\begin{array}{ll} a_0^5 \cdots & 2 \text{ rows} \\ b_0^5 \cdots & 2 \text{ rows} \\ \hline \alpha_0^{11} \cdots \alpha_6^{11} & \\ \alpha_1^{11} \cdots \alpha_7^{11} & \\ \beta_0^8 \cdots \beta_6^8 & \\ \gamma_0^5 \cdots \gamma_6^5 & \end{array}$$

The γ row is added by comparing the sets of degree 5, assuming the 2d b row dependent. From this it follows that the two independent arrays

$$\begin{array}{ll} a_0^5 \cdots & 1 \text{ row} \\ b_0^5 \cdots & 1 \text{ row} \\ \hline \alpha_0^{11} \cdots \alpha_5^{11} & \\ \alpha_1^{11} \cdots \alpha_6^{11} & \\ \beta_0^8 \cdots \beta_5^8 & \\ \gamma_0^5 \cdots \gamma_5^5 & \\ \gamma_1^5 \cdots \gamma_6^5 & \end{array}$$

correspond, which is impossible. Hence the set of degree 6 in the ζ array is regular. Similarly the rest of the proof of the direct theorem can be reversed; and it follows that the sets of degree 9, 11, 12 in the ζ array are regular. This proves the converse theorem. The one-set theorem is, I believe, true for any number of dimensions; but the converse is only true for two dimensions.

The irregularities of a one-set.

15. The diagram (T § 32) of the equations of the one-set of § 12 above is

$$\begin{array}{l|llll} z^{11} & E_0^0 & & & \\ z^{10} & E_0^1 & E_1^1 & & \\ z^9 & E_0^2 & E_1^2 & & \\ z^8 & E_0^3 & E_1^3 & F_0^0 & \\ z^7 & E_0^4 & E_1^4 & F_0^1 & \\ z^6 & E_0^5 & E_1^5 & F_0^2 & \\ z^5 & E_0^6 & E_1^6 & F_0^3 & G_0^0 \\ z^4 & E_0^7 & E_1^7 & F_0^4 & G_0^1 \quad G_1^1, \end{array}$$

where $E_0^0 = \Sigma\alpha_q^p z_q^p$, $F_0^0 = \Sigma\beta_q^p z_q^p$, $G_0^0 = \Sigma\gamma_q^p z_q^p$. This diagram shows the irregularities in the simplest way. There are three irregularities altogether, one before F_0^0 appears, and two consecutive ones before G_0^0 appears. Consecutive irregularities can be considered together, but separated irregularities must be considered by themselves in the order in which they appear.

It is clear that F_0^0 replaces E_2^2, and is equal to E_2^2 modified by E's which occur in the first two columns, viz.,

$$F_0^0 = f_0^2 E_0^2 + f_1^2 E_1^2 + f_2^2 E_2^2, \qquad \text{where} \qquad f_2^2 = 1.$$

We shall say that $x^{p-q}y^q$ corresponds to E_q^p, so that the polynomial which corresponds to F_0^0 is

$$F = f_0^2 x^2 + f_1^2 xy + f_2^2 y^2.$$

Again G_0^0 is F_1^1 (or E_3^3) modified, viz.,

$$G_0^0 = (F_0^1, F_1^1, F_0^2, E_0^4, E_1^4, E_0^5, E_1^5)^{(1)} = g_0^3 E_0^3 + \cdots,$$

and to G_0^0 corresponds a series $g(g_3^3 = 1)$, and polynomial G, viz.,

$$G = g_0^3 x^3 + g_1^3 x^2 y + g_2^3 xy^2 + g_3^3 y^3 + g_0^4 x^4 + g_1^4 x^3 y + g_2^4 x^2 y^2 + g_0^5 x^5 + g_1^5 x^4 y.$$

Also, from the first expression for G_0^0 above, since xF, yF correspond to F_0^1, F_1^1, we have

$$G = Fu_1 + P_4,$$

where u_n is a homogeneous polynomial in x, y of degree n, and P_n denotes a polynomial, or power series, whose lowest terms are of degree n.

16. The relation $f_0^2 E_0^2 + f_1^2 E_1^2 + f_2^2 E_2^2 = F_0^0$, where F_0^0 is one degree lower in z's than E_0^2, E_1^2, E_2^2, shows that the following arrays correspond:

	z_0^5	z_1^5	z_2^5	z_3^5	z_4^5	z_5^5
f_0^2	α_0^{11}	α_1^{11}	α_2^{11}	α_3^{11}	α_4^{11}	α_5^{11}
f_1^2	α_1^{11}	α_2^{11}	α_3^{11}	α_4^{11}	α_5^{11}	α_6^{11}
f_2^2	α_2^{11}	α_3^{11}	α_4^{11}	α_5^{11}	α_6^{11}	α_7^{11}
l	f_0^2	f_1^2	f_2^2			
m		f_0^2	f_1^2	f_2^2		
n			f_0^2	f_1^2	f_2^2	
r				f_0^2	f_1^2	f_2^2

Hence the lower array gives a complete solution for $z_0^5, \cdots, z_5^5$, so that

$$z_0^5 x^5 + z_1^5 x^4 y + \cdots + z_5^5 y^5$$
$$= (f_0^2 x^2 + f_1^2 xy + f_2^2 y^2)(lx^3 + mx^2 y + nxy^2 + ry^3) = Fu_3$$

or, if $S = \Sigma z_q^p x^{p-q} y^q$ is the general polynomial which satisfies the one-set,

$$S = Fu_3 + P_6.$$

Again, from the relation

$$g_0^3 E_0^3 + \cdots + g_3^3 E_3^3 + g_0^4 E_0^4 + g_1^4 E_1^4 + g_2^4 E_2^4 + g_0^5 E_0^5 + g_1^5 E_1^5 = G_0^9,$$

combined with the former relation, we see that the two following arrays correspond:

	$z_0^5 z_1^5 z_2^5 z_3^5 z_4^5 z_5^5$	$z_0^6 z_1^6 z_2^6 z_3^6 z_4^6 z_5^6 z_6^6$	$z_0^7 z_1^7 z_2^7 z_3^7 z_4^7 z_5^7 z_6^7 z_7^7$
g_0^3	$\alpha_0^9 \cdots \alpha_5^9$	$\alpha_0^{10} \cdots \alpha_6^{10}$	$\alpha_0^{11} \cdots \alpha_7^{11}$
g_1^3	$\alpha_1^9 \cdots \alpha_6^9$	$\alpha_1^{10} \cdots \alpha_7^{10}$	$\alpha_1^{11} \cdots \alpha_8^{11}$
g_2^3	$\alpha_2^9 \cdots \alpha_7^9$	$\alpha_2^{10} \cdots \alpha_8^{10}$	$\alpha_2^{11} \cdots \alpha_9^{11}$
g_3^3	$\alpha_3^9 \cdots \alpha_8^9$	$\alpha_3^{10} \cdots \alpha_9^{10}$	$\alpha_3^{11} \cdots \alpha_{10}^{11}$
g_0^4	$\alpha_0^{10} \cdots \alpha_5^{10}$	$\alpha_0^{11} \cdots \alpha_6^{11}$	
g_1^4	$\alpha_1^{10} \cdots \alpha_6^{10}$	$\alpha_1^{11} \cdots \alpha_7^{11}$	
g_2^4	$\alpha_2^{10} \cdots \alpha_7^{10}$	$\alpha_2^{11} \cdots \alpha_8^{11}$	
g_0^5	$\alpha_0^{11} \cdots \alpha_5^{11}$		
g_1^5	$\alpha_1^{11} \cdots \alpha_6^{11}$		

$g_0^3 g_1^3 g_2^3 g_3^3$	$g_0^4 g_1^4 g_2^4$	$g_0^5 g_1^5$	3 rows
	$g_0^3 g_1^3 g_2^3 g_3^3$	$g_0^4 g_1^4 g_2^4$	4 rows
		$f_0^2 f_1^2 f_2^2$	6 rows.

Hence

$$S = G(u_2 + u_3) + Fu_5 + P_8,$$

since the upper array has all its rows independent except the 4th, and consequently the lower array gives a complete solution for z^5, z^6, z^7.

Two results should be noted, which hold for any one-set: (i) S is expressible by means of F, G, $\cdots$ up to and including terms of a degree equal to the index of E in the last row of the diagram of the one-set; and (ii) the solution $S = G(u_2 + u_3) + Fu_5 + P_8$ is the general solution of the i equations of the one-set (and all their derivates) contained in the ith line of the diagram when written in E's only, viz., in the present example,

$$\begin{array}{lllll}
E_0^0 & & & & \\
E_0^1 & E_1^1 & & & \\
E_0^2 & E_1^2 & E_2^2 & & \\
E_0^3 & E_1^3 & E_2^3 & E_3^3 & \\
E_0^4 & E_1^4 & E_2^4 & E_3^4 & E_4^4 \\
E_0^5 & E_1^5 & E_2^5 & & \\
E_0^6 & E_1^6 & E_2^6 & & \\
E_0^7 & E_1^7. & & &
\end{array}$$

The general solution of the whole one-set is obtained by applying to u_2, u_3, u_5, P_8 the conditions which arise from the equations contained in the first 4 (or $i-1$) lines of this diagram. There are two equations ($F_0^1 = 0$, $G_0^0 = 0$) which involve the coefficients of u_2, u_3, u_5 only, of which one ($G_0^0 = 0$) involves the coefficients of u_2 only; the other 8 involve in addition the coefficients of P_8. The above form of the diagram is easily obtained from the original form by writing for the F's and G's their equivalent E's.

17. Take the still more complicated one-set whose diagram is

$$\begin{array}{lllllll}
z^{17} & E_0^0 & & & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
z^{13} & E_0^4 & E_1^4 & F_0^0 & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
z^7 & E_0^{10} & E_1^{10} & F_0^6 & F_1^6 & F_2^6 & G_0^0 \\
z^6 & E_0^{11} & E_1^{11} & F_0^7 & F_1^7 & F_2^7 & G_0^1 \\
z^5 & E_0^{12} & E_1^{12} & F_0^8 & F_1^8 & F_2^8 & G_0^2.
\end{array}$$

There are three groups of consecutive irregularities, to the third of which corresponds an expression H_0^0, containing no z^5 or z^6, where

$$H_0^0 = (G_0^1, G_1^1, G_0^2, F_0^7, F_1^7, F_2^7, F_0^8, F_1^8, F_2^8, E_0^{11}, E_1^{11}, E_0^{12}, E_1^{12})^{(1)}.$$

Since F_0^3, G_0^0, H_0^0 replace E_2^3, E_5^5, E_6^6 respectively, we have, as in § 16,

$$S = F(u_4 + u_5) + P_8,$$

$$S = G(u_1 + u_2 + u_3) + F(u_7 + u_8) + P_{11},$$

$$S = H(u_0 + u_1) + G(u_3 + u_4 + u_5) + F(u_9 + u_{10}) + P_{13}. \tag{1}$$

Also, from the relations,

$$G_0^0 = (F_0^3, F_1^3, F_2^3, F_3^3, \cdots, E_0^7, E_1^7, \cdots)^{(1)},$$

$$H_0^0 = (G_0^1, G_1^1, G_0^2, F_0^7, F_1^7, F_2^7, \cdots, E_0^{11}, E_1^{11}, \cdots)^{(1)},$$

we have

$$G = F(v_3 + v_4) + P_7, \tag{2}$$

$$H = G(v_1 + v_2) + F(v_7 + v_8) + P_{11}, \tag{3}$$

$$H = G(v_1 + v_2) + P_9, \qquad \text{from (3)}, \tag{4}$$

$$H = F(v_1 v_3 + v_5) + P_8, \qquad \text{from (2) and (3)}. \tag{5}$$

In (2) and (3) we may restrict v_4 to 3 terms, x^4, x^3y, x^2y^2, v_2 to 1 term, and v_7, v_8 each to 3 terms; but this has no special significance for what follows. None of the coefficients of u_0, u_1, u_3, u_4, u_5, u_9, u_{10}, in (1) can be dispensed with, although other forms than (1) might be adopted for S. The polynomial F contains terms of degree 2 and 3, G contains terms of degree 5 to 9, and H contains terms of degree 6 to 12. The relations (2) and (3) show that the addition of Hu_2 or Gu_6 would not alter the form of the right hand side of (1).

Geometrical interpretation. When the last of the polynomials F, G, $\cdots$ has all its tangents distinct the above results can be interpreted geometrically. The line E_0^1, E_1^1 of the diagram, being followed by 7 irregularities in all, indicates that the two branches of S whose tangents are $f_0^2x^2 + f_1^2xy + f_2^2y^2 = 0$ have contact of the 7th order with two branches of H, i. e., they are fixed to the 7th order. The line E_0^6, E_1^6, F_0^2, F_1^2, F_2^2, being followed by 5 irregularities, indicates that 3 more branches of S, whose tangents are $v_3 = 0$, have contact of the 5th order with three other branches of H; and the line E_0^{10}, $\cdots$, G_0^0, being followed by 2 irregularities, indicates that the last branch of S, whose tangent is $v_1 = 0$, has contact of the 2d order with the last branch of H. In other words, all the 6 branches of S are fixed to the 2d order, 5 branches are fixed to the 5th order, and 2 to the 7th order.

To prove this, let G', F' denote the power series which divide H, and have the same tangents as G, F respectively. Then from (4) $G = G' + P_8$, and from (1) $S = G'P_1 + P_{11}$; hence 5 of the branches of S have contact with G', and with H, of the 5th order. Again, from (5), $F = F' + P_4$, and from (3), $G(v_1 + v_2) = F'(v_1v_3 + \cdots) + P_{11}$; hence F' has contact with G of the 5th order, i. e., $G = F'(v_3 + \cdots) + P_{10}$; hence, from (1), $S = F'P_4 + P_{13}$; i. e., two of the branches of S have contact with F', and with H, of the 7th order.

This interpretation of the irregularities agrees with that suggested by Miss Scott in T § 32. The interpretation does not apply if H has any cusp or higher singularity at the origin.

18. *The number of intersections.* It follows from § 17, when the last of the polynomials F, G, $\cdots$ has all its tangents distinct, that the whole intersection at the origin of any two curves which satisfy the one-set is at least equivalent to as many ordinary points as there are independent equations in the one-set. From the diagram we can choose 6 lines containing 1, 2, $\cdots$, 6 equations respectively, to which may be added the 15 equations $z^p = 0$ when $p < 5$ (the equations $z^5 = 0$ being included in the diagram). These make 6^2 (or i^2) equations. There are still left in the diagram 2 lines containing 2 equations preceding the line containing F_0^0, 3 lines of 5 equations preceding G_0^0, and 2 lines of 6 equations after G_0^0; counting these by columns there are $2 \times 7 + 3 \times 5 + 1 \times 2$ equations, which equals the number of additional intersections of the two curves due to the contacts of their several branches, which they have in common with H. If the two curves had further contact the number of their intersections would exceed the number of independent equations of the one-set; but in this case it follows from what is proved below that the one-set would not then be the whole one-set of the two curves.

This result follows from the relations (1), (2), (3) on the supposition that all the tangents of H are distinct. If H had some or all of its tangents coincident, the number of the intersections would not thereby be diminished, although it is conceivable that it might be increased. Hence we may assume it to be always true that the number of intersections at the origin of any two curves which satisfy a given one-set is at least equal to the number of independent equations of the one-set.

From this it can be shown that the two numbers are exactly equal when the one-set is the whole one-set of the two curves. Choose two curves C_1, C_2 of degrees l, m, having a given one-set for their whole one-set (§ 14), and such that their terms of highest degree have no factor in common. Let S be a curve of degree $\geqq l + m - 2$ with all its coefficients at disposal. By making S satisfy the one-set, we have $S = C_1P_1 + C_2P_2$. Also, by moving the origin in succession to each point of intersection of C_1, C_2, and making S satisfy all the corresponding one-sets, S takes the form $C_1P_1 + C_2P_2$ for every point of intersection of C_1, C_2. Hence, by NOETHER's Theorem, $S = C_1S_1 + C_2S_2$, where C_1S_1 and C_2S_2 are of the same degree as S, since the terms of degree l, m in C_1, C_2 have no common factor. But the number of independent equations that must be satisfied by the coefficients of S (of degree $\geqq l + m - 2$) in order that it may be of the above form is known to be lm, which is also the number of ordinary points of intersection of C_1, C_2. Hence the equations of the one-sets of C_1, C_2 for all their points of intersection are equivalent to the same

number (lm) of independent equations. This can only be true when the number of independent equations in each one-set is equal to (since it does not exceed) the corresponding number of ordinary intersections.

19. Another theorem of importance follows from § 17. Any t-set contained in a one-set E can be written in the form of the equations $\Sigma a_q^p E_q^p = 0$, $\Sigma b_q^p E_q^p = 0, \cdots$ with all their derivates $\Sigma a_q^p E_{q+m}^{p+l} = 0, \cdots$. The theorem is that *if the t-set includes all the equations in the first line of the diagram of E which has the full complement of i equations then any series which satisfies the one-set is expressible by means of the series $a, b, \cdots$*, i. e.,

$$S = AP + BQ + \cdots$$

where $A = \Sigma a_q^p x^{p-q} y^q, \cdots$, *and* $P, Q, \cdots$ *are power series. Also* $P, Q, \cdots$ *can all be chosen so as to vanish when* $x = y = 0$. *Also, if* $\Sigma a_q'^p E_q^p = 0$ *is any equation included in the t-set, then the series a' is expressible by means of the series* $a, b, \cdots$.

We shall prove this for the one-set of § 17. The t-set then includes all the equations E_0^{10}, E_1^{10}, F_0^6, F_1^6, F_2^6, G_0^0. Any equation $\Sigma a_q'^p E_q^p = 0$ included in the t-set is expressible identically in the form

$$\Sigma a_q'^p E_q^p = \Sigma \lambda_s^r \Sigma a_q^p E_{q+s}^{p+r} + \Sigma \mu_s^r \Sigma b_q^p E_{q+s}^{p+r} + \cdots,$$

for $\Sigma a_q^p E_{q+s}^{p+r}$, etc., include all the independent expressions in z involved in the equations of the t-set. This identity, however, is one in z's; but we may equate coefficients of E's if we first add to the right hand side all expressions in E's which identically vanish, each with an undetermined multiplier. All such expressions, as well as several others which do not identically vanish, are included in x-derivates of E_0^{10}, E_1^{10}, F_0^6, F_1^6, F_2^6, and all derivates of G_0^0. Hence we obtain

$$A' = AP + BQ + \cdots + GP_1 + FP_7 + P_{11},$$

where P_n denotes as before a power series whose lowest terms are of degree n. It has to be shown that $GP_1 + FP_7 + P_{11}$ may be omitted from this identity. Since E_0^{10}, E_1^{10} are included in the t-set and E_2^{10}, E_3^{10}, $\cdots$, E_{10}^{10} are expressible in terms of these and F's with index $\geqq 8$ (for F_0^6 replaces E_2^2) we may put A' equal to any one of $x^{10}, x^9 y, \cdots, y^{10}$; hence P_{11} may be omitted from the form for A'. Again, since F_0^6, F_1^6, F_2^6 are included in the t-set, and $F_3^6, \cdots F_6^6$ are expressible in terms of these and E's with index $\geqq 10$ and G's with index $\geqq 3$, we may put A' equal to any one of the polynomials $x^6 F, x^5 yF, \cdots, y^6 F$; from which it easily follows that FP_7 may be omitted in the form for A'. Finally, since G_0^0 is included in the t-set, we may write G for A'; so that GP_1 may be omitted. Hence

$$A' = AP + BQ + \cdots,$$

i. e., the series a' is expressible by means of the series $a, b, \cdots$

Also, since (§ 17)

$$S = G(u_1 + u_2 + u_3) + F(u_7 + u_8) + P_{11},$$

and G, FP_6, P_{10} are all of the form A', we have

$$S = AP + BQ + \cdots,$$

where P, Q, $\cdots$ are power series which all vanish when $x = y = 0$. Hence also we have the following theorem, which will be used later: —

A one-set E can always be found containing any given t-set, such that if the t-set be expressed in any manner in the form $\Sigma a_q^p E_q^p = 0$, $\Sigma b_q^p E_q^p = 0$, $\cdots$, then any polynomial satisfying the one-set is of the form $AP + BQ + \cdots$, where $A = \Sigma a_q^p x^{p-q} y^q$, $\cdots$, and P, Q, $\cdots$ are power series which vanish when $x = y = 0$.

Let a', b', $\cdots$ be the series, and A', B', $\cdots$ the corresponding polynomials, by means of which the general solution of the t-set can be expressed. Let A' have terms of as low degree i as any one of the polynomials A', B', $\cdots$. Divide A' into all the other polynomials until remainders are obtained commencing with terms of as high degree as possible; these remainders may be taken as the polynomials B', C', $\cdots$. Let B' have terms of as low degree j ($\leqq i$) as any one of B', C', $\cdots$. Then the one-set of A', B' satisfies the conditions. This one-set contains the t-set; each of the lines in its diagram corresponding to z^{i-1}, $\cdots$, z^{j-1} contains i equations, and the line corresponding to z^j contains only $i - 1$ equations. Also the equations in the line corresponding to z^{j-1} gives all the one-sets of degree $j - 1$ satisfied by the series a'; and these one-sets are all satisfied by each of the series a', b', $\cdots$ (since every b', c', $\cdots$ with index $< j$ vanishes), and are therefore all included in the t-set. Hence the theorem follows.

It is not necessary for this theorem that the series a, b, $\cdots$ should be independent, or limited in number; nor is it necessary that the one-set should be restricted to the choice that has been made. It is proved in §§ 22, 23 that the one-set E may be chosen as the one-set of any pair of the series a', b', $\cdots$, provided these are all independent.

20. In the case where the ζ equations of two series a, b include only k regular sets altogether, where $k < i$, the following array will vanish, however far it may be carried:

$$\begin{array}{cccccccc}
a_0^i & \cdots & a_0^{i+1} & \cdots & a_0^{i+2} & \cdots & k+1 & \text{rows} \\
b_0^i & \cdots & b_0^{i+1} & \cdots & b_0^{i+2} & \cdots & k+1 & \text{“} \\
 & & a_0^i & \cdots & a_0^{i+1} & \cdots & k+2 & \text{“} \\
 & & b_0^i & \cdots & b_0^{i+1} & \cdots & k & \text{“} \\
 & & & & a_0^i & \cdots & k+3 & \text{“} \\
 & & & & b_0^i & \cdots & k & \text{“} \\
 & & & & & \cdots & & \\
 & & & & & \cdots & &
\end{array}$$

there being only k rows b in each set except the first. For if the rows of this array were all independent, and we modify it so that the leading elements in each set may be independent, we must come sometime to a set containing $k+1$ rows in addition to the rows a, which is contrary to the hypothesis. Hence the whole last row of the 1st set is dependent on the other rows of the array. We may border the array by a row of arbitrary elements, and a column which is not arbitrary, viz.,

$$\rho_0^k, \cdots, \rho_k^k, \sigma_0^k, \cdots, \sigma_k^k, \rho_0^{k+1}, \cdots, \rho_{k+1}^{k+1}, \sigma_0^{k+1}, \cdots, \sigma_{k-1}^{k+1},$$
$$\rho_0^{k+2}, \cdots, \rho_{k+2}^{k+2}, \sigma_0^{k+2}, \cdots, \sigma_{k-1}^{k+2}, \cdots,$$

in which σ_k^k is not zero, and take the sum of every element in the array multiplied by the two corresponding elements in the border to be zero. Choosing the elements of the border row to be

$$x^{i+k}, x^{i+k-1}y, \cdots, y^{i+k}, x^{i+k+1}, x^{i+k}y, \cdots, y^{i+k+1}, x^{i+k+2}, \cdots,$$

we obtain the identity

$$\rho_0^k x^k C_1 + \rho_1^k x^{k-1} y C_1 + \cdots + \sigma_0^k x^k C_2 + \sigma_1^k x^{k-1} y C_2 + \cdots = 0,$$

or

$$C_1 P_1 + C_2 P_2 = 0,$$

where P_1, P_2 are power series in which terms of order k are present, since $\sigma_k^k \neq 0$. Hence C_1, C_2 have a common factor with a multiple point at least of order $i-k$ at the origin;* for if all the common factors of P_2, P_1 which vanish at the origin be divided out, C_1 and C_2 will be divisible by the factors which remain, and the quotients will be identical, except in sign.

Independence of one-sets and of series.

21. *A t-set system of equations is not equivalent to any but a t-set system.* It will be sufficient to prove this for a 3-set, $E = E' = E'' = 0$. In the first place no identical relation of the form

$$\Sigma \lambda_n^r E_n^r + \Sigma \lambda_n^{r'} E_n^{r'} + \Sigma \lambda_n^{r''} E_n^{r''} = 0$$

can exist, in which $\lambda_0^0, \lambda_0^{r'}, \lambda_0^{r''}$ are not all zero. For if $\lambda_0^0 = 1$, then, by taking all the derivates of the identity we can find all the derivates of E, and finally E itself, in terms of E', E'' and their derivates, so that the system is not a 3-set. Suppose then that the 3-set is equivalent to a 2-set $F = F' = 0$. Then E, E', E'' are expressible in terms of F, F' and their derivates, and also F, F' in terms of E, E', E'' and their derivates, i. e.,

*BERRY, *On a case of divisibility, etc.*, Proceedings of the London Mathematical Society, vol. 30 (1899), p. 275.

$$E = \lambda F + \lambda' F' + \cdots + \lambda_s^r F_s^r + \lambda_s^{'r} F_s^{'r} + \cdots,$$

$$E' = \mu F + \mu' F' + \cdots,$$

$$E'' = \nu F + \nu' F' + \cdots,$$

$$F = \rho E + \rho' E' + \rho'' E'' + \cdots,$$

$$F' = \sigma E + \sigma' E' + \sigma'' E'' + \cdots.$$

Substitute from the last two and their derivates in the first three, then

$$E = \lambda(\rho E + \rho' E' + \rho'' E'') + \lambda'(\sigma E + \sigma' E' + \sigma'' E'') + \cdots,$$

$$E' = \mu(\rho E + \rho' E' + \rho'' E'') + \mu'(\sigma E + \sigma' E' + \sigma'' E'') + \cdots,$$

$$E'' = \nu(\rho E + \rho' E' + \rho'' E'') + \nu'(\sigma E + \sigma' E' + \sigma'' E'') + \cdots.$$

In the last three identities the coefficients of E, E', E'' vanish, from above; hence we have

$$\begin{vmatrix} \lambda & \lambda' & 0 \\ \mu & \mu' & 0 \\ \nu & \nu' & 0 \end{vmatrix} \times \begin{vmatrix} \rho & \sigma & 0 \\ \rho' & \sigma' & 0 \\ \rho'' & \sigma'' & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1,$$

which is impossible. Hence it is clear that the 3-set is not equivalent to any but a 3-set.

Any number of series $a, b, c, \cdots$ are said to be independent if no one is expressible by means of the rest. In such a case no relation of the form $\Sigma\alpha_s^r a_{q-s}^{p-r} + \Sigma\beta_s^r b_{q-s}^{p-r} + \cdots = 0$ can exist for all values of p, q $(p \geqq q \geqq 0)$ fixed for one relation, unless $\alpha_0^0, \beta_0^0, \cdots$ all vanish. For the above relation gives an identity $A_1 C_1 + A_2 C_2 + \cdots = 0$, where $A_1, A_2, \cdots, C_1, C_2, \cdots$ are the power series corresponding to the series $\alpha, \beta, \cdots, a, b, \cdots$, from which one of the power series $C_1, C_2, \cdots$ is expressible by means of the rest if one of the constant terms of $A_1, A_2, \cdots$, viz., $\alpha_0^0, \beta_0^0, \cdots$, does not vanish. From this it can be proved in the same way as above that $t+1$ *independent series are not equivalent to any but* $t+1$ *independent series either in respect to the series which are expressible by means of them, or in respect to the one-sets which they satisfy.*

The t-set theorem.

22. Consider all the one-sets satisfied by the $t'+2$ given independent series $a', b', \cdots$. Let $E = \Sigma\alpha_q^p z_q^p$ be any one-set (e. g., the one-set of a', b', which is the choice we shall eventually make) containing all the one-sets. Then the general one-set contained in E is

$$\Sigma z_q^{'p} E_q^p = 0.$$

In order that this and all its derivates $\Sigma z_q'^p E_{q+m}^{p+l} = 0$ may be satisfied by the series a', we have

$$\Sigma z_q'^p A_q'^p = 0, \ \cdots, \ \Sigma z_q'^p A_{q+m}'^{p+l} = 0, \ \cdots,$$

where $A_q'^p$ is the value of E_q^p when the series a' is written for z. But

$$\Sigma z_q'^p A_q'^p = \Sigma z_q'^p \Sigma \alpha_s^r a_{s-q}'^{r-p} = \Sigma z_q'^p \Sigma \alpha_{s+q}^{r+p} a_s'^r$$

$$= \Sigma a_s'^r \Sigma \alpha_{s+q}^{r+p} z_q'^p = \Sigma a_s'^r \Sigma \alpha_q^p z_{q-s}'^{p-r} = \Sigma a_s'^r E_s'^r,$$

and similarly

$$\Sigma z_q'^p A_{q+m}'^{p+l} = \Sigma a_s'^r E_{s+m}'^{r+l}.$$

Hence the necessary and sufficient condition that the series a' may satisfy the one-set $\Sigma z_q'^p E_q^p = 0$ is that the series z' should satisfy the one-set $\Sigma a_s'^r E_s^r = 0$, or $\Sigma a_q'^p E_q^p = 0$; and in order that all the series $a', b', \cdots$ may satisfy the one-set $\Sigma z_q'^p E_q^p = 0$, it is necessary and sufficient that the series z' should satisfy the $t' + 2$ one-sets $\Sigma a_q'^p E_q^p = 0$, $\Sigma b_q'^p E_q^p = 0, \ \cdots$.

Let the general solution of these $t' + 2$ one-sets be expressible by the t independent series $a, b, \cdots$; then expressing z' by the series $a, b, \cdots$, and substituting in $\Sigma z_q'^p E_q^p = 0$, we see that the t one-sets

$$\Sigma a_q^p E_q^p = 0, \ \Sigma b_q^p E_q^p = 0, \ \cdots \qquad (t \text{ one-sets})$$

make up all the one-sets which the given series $a', b', \cdots$ satisfy. Between these t one-sets, and the $t' + 2$ one-sets

$$\Sigma a_q'^p E_q^p = 0, \ \Sigma b_q'^p E_q^p = 0, \ \cdots \qquad (t' + 2 \text{ one-sets}),$$

a reciprocal relation exists, viz., the complete solution of the t one-sets is given by the $t' + 2$ independent series $a', b', \cdots$, and the complete solution of the $t' + 2$ one-sets is given by the t independent series $a, b, \cdots$. Such mutually related systems are said to be residual with respect to the one-set E.*

* If S is the general power series which satisfies the t one-sets, then

$$S = A'P + B'Q + \cdots,$$

where $A', B', \cdots$ are the power series corresponding to the series $a', b', \cdots$, and $P, Q, \cdots$ are arbitrary power series. Further, in order that S' may be the general power series such that SS' satisfies the one-set, it is necessary and sufficient that $A'S', B'S', \cdots$ should satisfy the one-set, i. e., that S' should satisfy the $t' + 2$ one-sets, or that S' should be of the form

$$S' = AP + BQ + \cdots.$$

Thus, if S, S' are any power series satisfying the t one-sets and $t' + 2$ one-sets respectively, then SS' satisfies the one-set. It is proved in P § 32 that the number of independent equations in the t one-sets added to the number of independent equations in the $t' + 2$ one-sets is equal to the number of independent equations in the one-set E.

23. The $t'+2$ one-sets $\Sigma a'^{p}_{q} E^{p}_{q} = 0$, etc., will be independent provided the series h, k which determine the one-set E are such that

$$H = A'P + B'Q + \cdots,$$
$$K = A'P' + B'Q' + \cdots,$$

where $P, Q, \cdots, P', Q', \cdots$ are power series which all vanish when $x = y = 0$. For the $t'+1$ one-sets $\Sigma b'^{p}_{q} E^{p}_{q} = 0$, $\Sigma c'^{p}_{q} E^{p}_{q} = 0, \cdots$ are the equations which the series z' has to satisfy in order that the series $b', c', \cdots$ may satisfy the one-set $\Sigma z'^{p}_{q} E^{p}_{q} = 0$; and if the one-set $\Sigma a'^{p}_{q} E^{p}_{q} = 0$ is dependent on the remaining $t'+1$ one-sets, the series a' must satisfy all the one-sets contained in E, viz., all the one-sets $\Sigma z'^{p}_{q} E^{p}_{q} = 0$, which the series $b', c', \cdots$ satisfy, i. e., a' must satisfy all the one-sets which $h, k, b', c', \cdots$ satisfy. Hence A' must be expressible by means of $H, K, B', C', \cdots$ and therefore also by means of $B', C', \cdots$ alone, which is impossible, since the series $a', b', \cdots$ are independent.

Again, the $t'+2$ one-sets cannot reduce to less than t' independent one-sets, and do reduce to a t'-set if $H = A'$ and $K = B'$; for then $\Sigma a'^{p}_{q} E^{p}_{q}$ and $\Sigma b'^{p}_{q} E^{p}_{q}$ vanish identically. If the first three of the $t'+2$ one-sets were dependent on the rest, it would follow as above that the series a', b', c' would be expressible by means of h, k and the remaining series, so that the $t'+2$ independent series $a', b', \cdots$ would be equivalent to $t'+1$ independent series only, which is impossible (§ 20).

Similar properties hold for the t one-sets $\Sigma a^{p}_{q} E^{p}_{q} = 0$, etc., which form a definite or given system of equations, independent of E, since their solutions are the given series $a', b', \cdots$. Choose two solutions h, k of the t one-sets for determining the one-set E as in § 18, so that

$$H = AP + BQ + \cdots, \qquad K = AP' + BQ' + \cdots,$$

where $P, Q, \cdots, P', Q', \cdots$ all vanish when $x = y = 0$. In this case we may take H, K to be A', B' respectively; the t one-sets are then independent, from above, i. e., they make a t-set, and the $t'+2$ one-sets make a t'-set. Hence to any t-set, determined by $t'+2$ independent series, there corresponds a residual t'-set determined by t independent series. Thus, starting from the t-set, we can form the following chain of residues, to each of which we can assign the corresponding number of independent series,

t-set	t'-set	$(t-2)$-set	$(t'-2)$-set	etc.
$(t'+2)$ series	t series	t' series	$(t-2)$ series	

If t' is odd we must come to a one-set, to which nothing but two series can correspond (§ 14); and if t' is even, we must come to two series, to which noth-

ing but a one-set can correspond (§ 13). Hence it follows that $t = t' + 1$, and that all the one-sets which $t + 1$ independent series satisfy make up a t-set. This is the t-set theorem.

24. If the t independent one-sets, or t-set, $\Sigma a_q^p E_q^p = 0$, etc. and the t' independent one-sets, or t'-set, $\Sigma a_q'^p E_q^p = 0$, etc., are residual with respect to the one-set E, determined by the two series h, k, then the relations between the series $a, b, \cdots, a', b', \cdots$, and h, k are as follows. No one of the series $a, b, \cdots$ is expressible by means of the rest and h, k, since the t one-sets $\Sigma a_q^p E_q^p = 0$, etc., are independent. If h (or k) is dependent on $a, b, \cdots$, then, when H (or K) is expressed in the form $AP + BQ + \cdots$, the constant terms of the power series $P, Q, \cdots$ must all vanish. Similarly for the series $a', b', \cdots$, and h, k. The numbers t, t' can differ at most by unity.

(i) If $t' = t - 1$, the series h, k are dependent on $a, b, \cdots$, and independent of $a', b', \cdots$. This is the case of § 23 after omitting a', b'.

If $t' = t + 1$, the series h, k are dependent on $a', b', \cdots$, and independent of $a, b, \cdots$.

(ii) If $t' = t$, the series h, k can be so modified, without altering the whole one-set E determined by them, that h is dependent on $a, b, \cdots$ and independent of $a', b', \cdots$, and k is dependent on $a', b', \cdots$ and independent of $a, b, \cdots$. This is not proved above.

Also if a t-set and t'-set are residual with respect to a one-set E they can be expressed as $t' + 1$ one-sets and $t + 1$ one-sets respectively,

$$\Sigma a_q^p E_q^p = 0, \ \Sigma b_q^p E_q^p = 0, \ \cdots \quad (t' + 1 \text{ one-sets or } t\text{-set}),$$

$$\Sigma a_q'^p E_q^p = 0, \ \Sigma b_q'^p E_q^p = 0, \ \cdots \quad (t + 1 \text{ one-sets or } t'\text{-set}),$$

where the $t' + 1$ independent series $a, b, \cdots$ give the complete solution of the t'-set, and the $t + 1$ independent series $a', b', \cdots$ give the complete solution of the t-set. Here two of the $t' + 1$ one-sets are dependent on the rest, or two of the $t + 1$ one-sets are dependent on the rest, or one one-set is dependent on the rest in each system; each of the series h, k being among one or other of the $t + 1$ and $t' + 1$ series.

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ON CERTAIN COMPLETE SYSTEMS OF QUATERNION EXPRESSIONS, AND ON THE REMOVAL OF METRIC LIMITATIONS FROM THE CALCULUS OF QUATERNIONS*

BY

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§ 1. Every student of quaternions is familiar with the use, in connection with a system of three non-complanar vectors, α, β, γ, of a supplementary system of derived vectors, α', β', γ', which I shall here determine by the equations:

$$(1) \qquad \alpha' = \frac{V\beta\gamma}{S\gamma\beta\alpha}, \qquad \beta' = \frac{V\gamma\alpha}{S\gamma\beta\alpha}, \qquad \gamma' = \frac{V\alpha\beta}{S\gamma\beta\alpha}.$$

These equations give:

$$(2) \qquad \begin{cases} S\alpha\alpha' = S\beta\beta' = S\gamma\gamma' = -1, \\ S\alpha\beta' = S\alpha\gamma' = S\beta\gamma' = S\beta\alpha' = S\gamma\alpha' = S\gamma\beta' = 0, \\ S\alpha\beta\gamma S\alpha'\beta'\gamma' = 1, \\ \alpha = \dfrac{V\beta'\gamma'}{S\gamma'\beta'\alpha'}, \qquad \beta = \dfrac{V\gamma'\alpha'}{S\gamma'\beta'\alpha'}, \qquad \gamma = \dfrac{V\alpha'\beta'}{S\gamma'\beta'\alpha'}. \end{cases}$$

Thus α, β, γ can be derived from α', β', γ', just as α', β', γ' are derived from α, β, γ; and all the relations between these two sets of vectors are reciprocal.

The above equations give, for any vector ρ, the two identical expressions:

$$(3) \qquad \rho = -(\alpha S\alpha'\rho + \beta S\beta'\rho + \gamma S\gamma'\rho) = -(\alpha' S\alpha\rho + \beta' S\beta\rho + \gamma' S\gamma\rho).$$

It is to be noted that α', β', γ' are the negatives of the reciprocals of the vector perpendiculars dropped from the origin O on planes which pass through the points A, B, C, parallel to the planes OBC, OCA, OAB, respectively, and, in like manner, α, β, γ are the negatives of the reciprocals of the vector perpendiculars dropped from O on planes passing through A', B', C', parallel to OB'C', OC'A',

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OA′B′. Moreover, the volumes of the tetrahedra OABC and OA′B′C′ are reciprocal to each other.

§ 2. The six vectors $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$, may be said to constitute a *complete system* of vectors. I wish to point out the existence of other complete systems both of vectors and of quaternions, and also to propose certain modifications of the quaternion method, which are closely related to such systems, and which tend to enlarge the freedom of quaternion reasoning.

It is often assumed that the calculus of Quaternions is, in its relation to geometry, essentially a metric, as distinguished from a graphic, or projective, system. But this character seems to me to belong merely to the forms under which it is usually presented, and not to be inherent in the calculus itself. I shall show that it is easy to introduce into quaternions the principle of the *dualism of points and planes* familiar in modern analysis, and the principle of *homogeneousness*, which gives so great an advantage in projective geometry to tetrahedral over Cartesian coördinates, also to assume arbitrarily *any four linearly independent quaternions* as the fundamental geometric elements, and thus entirely to discard from our system all metric ideas.

§ 3. If $\lambda = \text{OL}$, where O is an assumed origin, we are accustomed to call λ "the vector of L" and L "the point λ." I purpose to employ the same symbol λ to denote also the *plane* which is polar to L relatively to the unit-sphere ($\rho^2 = -1$), having its center at the origin. Thus, a plane is to be denoted by the negative of the reciprocal of the vector perpendicular dropped on the plane from the origin; and we shall call any vector λ the vector of the plane which it represents in this system, as well as of the pole of that plane, and shall call the plane "the plane λ." The same vector symbol will therefore admit two interpretations. In this system, α', β', γ' defined as in § 1, are the vectors of the points A′, B′, C′, and also of the planes drawn through A, B, C, respectively, parallel to the opposite faces of the tetrahedron OABC; while α, β, γ admit corresponding double interpretations. In this system, again, $\lambda = 0$ for the origin and for the plane infinity; $\lambda = \infty$ for any point at infinity and for any plane through the origin. It will be shown later that infinites and indeterminates can be avoided by a further modification of the system (see § 8).

The condition of the *collocation* of a point ρ and a plane ω, that is, the condition of the point lying in the plane, or of the plane passing through the point, is:

$$\text{S}\omega\rho = -1. \tag{4}$$

This equation, which represents *any* linear scalar equation in ρ or in ω, is the equation of the plane ω, when ω is constant; of the point ρ, when ρ is constant; and when neither ω nor ρ is constant, of *any collocated point and plane.*

The relation between ρ and ω is analogous to that between the coördinates of a collocated point and plane in the ordinary analysis. This is clearly shown by writing, in terms of the complete system of § 1,

$$\begin{cases} \rho = x\alpha + y\beta + z\gamma, \\ \omega = u\alpha' + v\beta' + w\gamma'; \end{cases} \tag{5}$$

whence (4) gives at once

$$ux + vy + wz = 1, \tag{6}$$

which represents the Cartesian equation of the collocation of a point and a plane; their coördinates being in fixed ratios to x, y, z, u, v, w.

§ 4. HAMILTON has pointed out in the *Elements*, Book I, chap. iii, § 3, the advantage which may sometimes be gained by the use of *four* vectors, instead of three, for determining position; and has indicated the relation of such a system to tetrahedral (which he calls anharmonic) coördinates, and its adaptation to the expression of double ratios. But his treatment of the matter is inadequate; because he has not employed vectors dualistically, to represent both points and planes, and has failed to make his system of vectors complete by the introduction of four complementary vectors. The complete system may be formed as follows:

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be any four vectors, not termino-complanar, determining therefore the four vertices of a tetrahedron $A_1A_2A_3A_4$. Let us adopt the notation

$$\begin{cases} a_1 = \mathrm{S}\alpha_4\alpha_3\alpha_2, \quad a_2 = \mathrm{S}\alpha_3\alpha_4\alpha_1, \quad a_3 = \mathrm{S}\alpha_2\alpha_1\alpha_4, \quad a_4 = \mathrm{S}\alpha_1\alpha_2\alpha_3, \\ A = a_1 + a_2 + a_3 + a_4. \end{cases} \tag{7}$$

By a well-known identity, which is equivalent to (3),

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = 0, \tag{8}$$

and since $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are not termino-complanar,

$$A \neq 0. \tag{9}$$

Moreover, if we make the assumption that the origin (which is otherwise unlimited in position) lies in neither face of the tetrahedron, none of the quantities a_1, a_2, a_3, a_4, vanishes.

Let us now assume the vectors of the four faces of the fundamental tetrahedron as our four complementary vectors, $\beta_1, \beta_2, \beta_3, \beta_4$. We have then

$$\begin{cases} \beta_1 = \dfrac{\mathrm{V}(\alpha_2\alpha_3 + \alpha_3\alpha_4 + \alpha_4\alpha_2)}{a_1}, \quad \beta_3 = \dfrac{\mathrm{V}(\alpha_4\alpha_1 + \alpha_1\alpha_2 + \alpha_2\alpha_4)}{a_3}, \\ \beta_2 = \dfrac{\mathrm{V}(\alpha_1\alpha_4 + \alpha_4\alpha_3 + \alpha_3\alpha_1)}{a_2}, \quad \beta_4 = \dfrac{\mathrm{V}(\alpha_3\alpha_2 + \alpha_2\alpha_1 + \alpha_1\alpha_3)}{a_4}. \end{cases} \tag{10}$$

Computing hence the values of

$$(11)\quad \begin{cases} b_1 = \mathrm{S}\beta_4\beta_3\beta_2, \quad b_2 = \mathrm{S}\beta_3\beta_4\beta_1, \quad b_3 = \mathrm{S}\beta_2\beta_1\beta_4, \quad b_4 = \mathrm{S}\beta_1\beta_2\beta_3, \\ B = b_1 + b_2 + b_3 + b_4, \end{cases}$$

we readily find that

$$(12)\quad \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \frac{b_4}{a_4} = \frac{B}{A} = \frac{A^2}{a_1a_2a_3a_4} = \frac{b_1b_2b_3b_4}{B^2},$$

whence

$$(13)\quad A^2B^2 = a_1b_1a_2b_2a_3b_3a_4b_4;$$

and we obtain also the results

$$(14)\quad \begin{cases} \mathrm{S}\alpha_1\beta_1 = \dfrac{A}{a_1} - 1 = \dfrac{B}{b_1} - 1, \qquad \mathrm{S}\alpha_3\beta_3 = \dfrac{A}{a_3} - 1 = \dfrac{B}{b_3} - 1, \\ \mathrm{S}\alpha_2\beta_2 = \dfrac{A}{a_2} - 1 = \dfrac{B}{b_2} - 1, \qquad \mathrm{S}\alpha_4\beta_4 = \dfrac{A}{a_4} - 1 = \dfrac{B}{b_4} - 1, \\ \mathrm{S}\alpha_i\beta_j = -1, \text{ whenever } i \neq j. \end{cases}$$

From the symmetry of these equations, or by actual computation, we see that the α's are related to the β's just as the β's are related to the α's; so that if the β's are taken as the vertices of a tetrahedron $\mathrm{B_1B_2B_3B_4}$, the α's are the vectors of the faces of that tetrahedron, and may be expressed in terms of the β's by forms exactly analogous to (10).

§ 5. Since any five vectors satisfy one linear equation in which the sum of the coefficients is zero, and only one such equation unless all the vectors are termino-complanar, it follows that any vectors ρ and ω may be written in the forms

$$(15)\quad \begin{cases} \rho = \dfrac{m_1x_1\alpha_1 + m_2x_2\alpha_2 + m_3x_3\alpha_3 + m_4x_4\alpha_4}{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4}, \\ \omega = \dfrac{n_1u_1\beta_1 + n_2u_2\beta_2 + n_3u_3\beta_3 + n_4u_4\beta_4}{n_1u_1 + n_2u_2 + n_3u_3 + n_4u_4}, \end{cases}$$

where the four m's and the four n's are scalar constants arbitrarily chosen, provided none is taken equal to zero, while the x's and u's are scalar variables, of which the ratios in each of the two expressions are singly determinate for fixed values of ρ and ω.

Writing the first of equations (15) in the form

$$m_1x_1(\rho - \alpha_1) + m_2x_2(\rho - \alpha_2) + m_3x_3(\rho - \alpha_3) + m_4x_4(\rho - \alpha_4) = 0,$$

and observing that this equation is unique, we see, by the principle represented by equation (8), that

$$m_1x_1 : m_2x_2 : m_3x_3 : m_4x_4 = \mathrm{S}(\rho-\alpha_4)(\rho-\alpha_3)(\rho-\alpha_2) : \mathrm{S}(\rho-\alpha_3)(\rho-\alpha_4)(\rho-\alpha_1)$$

$$(16) \qquad : \mathrm{S}(\rho-\alpha_2)(\rho-\alpha_1)(\rho-\alpha_4) : \mathrm{S}(\rho-\alpha_1)(\rho-\alpha_2)(\rho-\alpha_3)$$

$$= \text{tet } \mathrm{A}_4\mathrm{A}_3\mathrm{A}_2\mathrm{R} : \text{tet } \mathrm{A}_3\mathrm{A}_4\mathrm{A}_1\mathrm{R} : \text{tet } \mathrm{A}_2\mathrm{A}_1\mathrm{A}_4\mathrm{R} : \text{tet } \mathrm{A}_1\mathrm{A}_2\mathrm{A}_3\mathrm{R},$$

where R is the point ρ. Hence, the x's may be identified with the tetrahedral coördinates of R referred to $\mathrm{A}_1\mathrm{A}_2\mathrm{A}_3\mathrm{A}_4$.

Again, the perpendiculars dropped from A_1, A_2, A_3, A_4, on the plane ω are in the continued proportion

$$(17) \quad \mathrm{S}\omega\alpha_1 + 1 : \mathrm{S}\omega\alpha_2 + 1 : \mathrm{S}\omega\alpha_3 + 1 : \mathrm{S}\omega\alpha_4 + 1 = \frac{n_1u_1}{a_1} : \frac{n_2u_2}{a_2} : \frac{n_3u_3}{a_3} : \frac{n_4u_4}{a_4};$$

so that the u's may be identified with the tetrahedral coördinates of the plane ω referred to $\mathrm{A}_1\mathrm{A}_2\mathrm{A}_3\mathrm{A}_4$.

Substituting from (15) in $\mathrm{S}\omega\rho = -1$, the equation of collocation of the point ρ and the plane ω, we have, by easy reductions,

$$\frac{m_1n_1}{a_1}u_1x_1 + \frac{m_2n_2}{a_2}u_2x_2 + \frac{m_3n_3}{a_3}u_3x_3 + \frac{m_4n_4}{a_4}u_4x_4 = 0,$$

or, if the m's and n's are so taken that

$$(18) \qquad m_1n_1 : m_2n_2 : m_3n_3 : m_4n_4 = a_1 : a_2 : a_3 : a_4,$$

$$(19) \qquad u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0,$$

the usual form of the equation of collocation of a point and a plane in tetrahedral coördinates.

§ 6. The dualism between points and planes can be further exhibited by the consideration of some familiar relations between vectors.

The condition of the collinearity of two vectors is that there is an equation of the form

$$(20) \qquad x_1\lambda_1 + x_2\lambda_2 = 0, \qquad \text{or that} \qquad \mathrm{V}\lambda_1\lambda_2 = 0;$$

that of the complanarity of three vectors is that there is an equation of the form

$$(21) \qquad x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3 = 0, \qquad \text{or that} \qquad \mathrm{S}\lambda_1\lambda_2\lambda_3 = 0;$$

while any four vectors in space satisfy an equation of the form

$$(22) \qquad \begin{cases} x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3 + x_4\lambda_4 = 0, & \text{in which} \\ x_1 : x_2 : x_3 : x_4 = \mathrm{S}\lambda_2\lambda_3\lambda_4 : \mathrm{S}\lambda_1\lambda_4\lambda_3 : \mathrm{S}\lambda_4\lambda_1\lambda_2 : \mathrm{S}\lambda_3\lambda_2\lambda_1. \end{cases}$$

If the λ's are vectors of points, (20) expresses the condition that the line joining two points passes through the origin; (21) the condition that the plane containing three points passes through the origin; while (22) is true for any

four points and is, indeed, the same as (8). If the λ's are vectors of planes, (20) expresses the condition of the parallelism of two planes, that is, that their common line lies in the plane infinity; (21) expresses the condition that three planes are parallel to one line, that is, that their common point lies in the plane infinity; while (22) is true for any four planes.

If, now, (20) is accompanied by the further condition

$$(23) \qquad x_1 + x_2 = 0, \qquad \text{which gives} \qquad \lambda_1 = \lambda_2,$$

the two points or two planes are coincident.

If (21) is accompanied by the condition

$$(24) \quad x_1 + x_2 + x_3 = 0, \qquad \text{which gives} \qquad \mathrm{V}(\lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2) = 0,$$

we have the condition that three points lie in one line or that three planes intersect in one line; since we easily deduce from (21) and (24) that for any value of σ,

$$(25) \qquad x_1(\mathrm{S}\lambda_1\sigma + 1) + x_2(\mathrm{S}\lambda_2\sigma + 1) + x_3(\mathrm{S}\lambda_3\sigma + 1) \equiv 0;$$

whence it follows that all values of σ which satisfy two of the three equations,

$$(26) \qquad \mathrm{S}\lambda_1\sigma + 1 = 0, \qquad \mathrm{S}\lambda_2\sigma + 1 = 0, \qquad \mathrm{S}\lambda_3\sigma + 1 = 0,$$

also satisfy the third; and conversely, if this be the case there must be an identity (25), whence (21) and (24) are easily obtained. If then the λ's denote three points, and σ a plane, every plane which contains two of the three points contains also the third, so that the three points are collinear, and conversely; but if the λ's denote three planes, and σ a point, every point belonging to two of the three planes lies also in the third, so that the three planes intersect in one line, and conversely.

It is shown, in like manner, that if, in (22),

$$(27) \quad x_1 + x_2 + x_3 + x_4 = 0, \quad \text{that is, if} \quad \mathrm{S}(\lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_4\lambda_3 + \lambda_4\lambda_1\lambda_2 + \lambda_3\lambda_2\lambda_1) = 0,$$

we have the condition of four points lying in one plane, or of four planes intersecting in one point.

The theorems of this section can also be obtained more briefly. For those which concern points follow immediately from the well-known propositions concerning linear equations between vectors; and those which concern planes can then be deduced from the others by the reciprocal relations between poles and polars.

§ 7. We pass next from complete systems of vectors to those of quaternions. Let p_1, p_2, p_3, p_4 be any four linearly independent quaternions; and let q_1, q_2, q_3, q_4 be four quaternions such that

$$(28) \quad \mathrm{S}p_1q_1 = \mathrm{S}p_2q_2 = \mathrm{S}p_3q_3 = \mathrm{S}p_4q_4 = -1, \quad \text{while} \quad \mathrm{S}p_iq_j = 0, \text{ if } i \neq j.$$

The four q's exist, and constitute a unique system; for each is determined by four linearly independent scalar equations of the first degree. Indeed, we know, by the theory of determinants of quaternions, that, using the notation

$$(29)\qquad \begin{cases} \bar{p}_1 = \mathrm{K}p_1, \qquad \bar{p}_2 = \mathrm{K}p_2, \quad \text{etc.}, \\ R_1 = \frac{1}{6}\,|\,\bar{p}_4 p_3 \bar{p}_2\,|, \quad R_2 = \frac{1}{6}\,|\,\bar{p}_3 p_4 \bar{p}_1\,|, \quad \text{etc.}, \\ R = \frac{1}{24}\,|\,p_1 \bar{p}_2 p_3 \bar{p}_4\,|\,; \end{cases}$$

$$(30)\qquad q_1 = \frac{R_1}{R}, \qquad q_2 = \frac{R_2}{R}, \qquad q_3 = \frac{R_3}{R}, \qquad q_4 = \frac{R_4}{R}.$$

Writing $p_1 = c_1 + \alpha_1$, $p_2 = c_2 + \alpha_2$, etc., we may express these forms as follows:

$$(31)\qquad \begin{cases} q_1 = \dfrac{\mathrm{S}\alpha_4\alpha_3\alpha_2 + (c_2\mathrm{V}\alpha_3\alpha_4 + c_3\mathrm{V}\alpha_4\alpha_2 + c_4\mathrm{V}\alpha_2\alpha_3)}{c_1\mathrm{S}\alpha_4\alpha_3\alpha_2 + c_2\mathrm{S}\alpha_1\alpha_4\alpha_3 + c_3\mathrm{S}\alpha_4\alpha_1\alpha_2 + c_4\mathrm{S}\alpha_3\alpha_2\alpha_1}, \\ q_2 = \dfrac{\mathrm{S}\alpha_3\alpha_4\alpha_1 + (c_1\mathrm{V}\alpha_4\alpha_3 + c_4\mathrm{V}\alpha_3\alpha_1 + c_3\mathrm{V}\alpha_1\alpha_4)}{c_1\mathrm{S}\alpha_2\alpha_3\alpha_4 + c_2\mathrm{S}\alpha_1\alpha_4\alpha_3 + c_3\mathrm{S}\alpha_4\alpha_1\alpha_2 + c_4\mathrm{S}\alpha_3\alpha_2\alpha_1}, \\ q_3 = \dfrac{\mathrm{S}\alpha_2\alpha_1\alpha_4 + (c_4\mathrm{V}\alpha_1\alpha_2 + c_1\mathrm{V}\alpha_2\alpha_4 + c_2\mathrm{V}\alpha_4\alpha_1)}{c_1\mathrm{S}\alpha_2\alpha_3\alpha_4 + c_2\mathrm{S}\alpha_1\alpha_4\alpha_3 + c_3\mathrm{S}\alpha_4\alpha_1\alpha_2 + c_4\mathrm{S}\alpha_3\alpha_2\alpha_1}, \\ q_4 = \dfrac{\mathrm{S}\alpha_1\alpha_2\alpha_3 + (c_3\mathrm{V}\alpha_2\alpha_1 + c_2\mathrm{V}\alpha_1\alpha_3 + c_1\mathrm{V}\alpha_3\alpha_2)}{c_1\mathrm{S}\alpha_2\alpha_3\alpha_4 + c_2\mathrm{S}\alpha_1\alpha_4\alpha_3 + c_3\mathrm{S}\alpha_4\alpha_1\alpha_2 + c_4\mathrm{S}\alpha_3\alpha_2\alpha_1}. \end{cases}$$

It is evident, moreover, from the symmetry of the equations (28) that the p's may be derived from the q's by the same rule by which the q's are derived from the p's. It is evident also that, since the p's are linearly independent, none of the above written determinants vanishes.

It is now easily seen that any quaternion r can be expressed identically in either of the following forms:

$$(32)\qquad \begin{cases} r = -(p_1\mathrm{S}q_1 r + p_2\mathrm{S}q_2 r + p_3\mathrm{S}q_3 r + p_4\mathrm{S}q_4 r) \\ \quad = -(q_1\mathrm{S}p_1 r + q_2\mathrm{S}p_2 r + q_3\mathrm{S}p_3 r + q_4\mathrm{S}p_4 r). \end{cases}$$

These identities correspond to the two identities (3) for any vector ρ; and they may also be written

$$(33)\qquad |\,p_1\bar{p}_2 p_3 \bar{r}\,| = |\,q_1\bar{q}_2 q_3\bar{r}\,| = 0;$$

just as (3) may be written

$$(34)\qquad |\,\alpha\beta\gamma\rho\,| = |\,\alpha'\beta'\gamma'\rho\,| = 0.$$

§ 8. I have shown that the principle of dualism can be introduced into quaternion analysis by using one and the same vector symbol to denote either a point or a plane. But in order to free our calculus entirely from the restrictions of metric geometry, we need also the principle of homogeneousness, whereby

we shall be able to dispense with infinites and indeterminates. This is accomplished by representing a point or a plane by a *quaternion*, and not by a mere vector. We shall at first assume for this purpose a quaternion such that the ratio of its vector to its scalar shall be the vector of the point or plane in the system expounded in § 3. Thus, the quaternion of a point or a plane shall have an undetermined scalar factor m, so that p and mp represent the same point or plane, so long as m is scalar. In this system, when the quaternion of a point is any vector, the point is at infinity, in the direction of that vector; when it is any scalar, the point is at the origin. When the quaternion of a plane is any vector, the plane passes through the origin, perpendicular to the vector; when any scalar, it is the plane infinity. Every equation which expresses a true geometric relation is necessarily homogeneous, in this system. The equation of the unit-sphere having its center at the origin and the equation of collocation of a point and plane are respectively:

$$\mathrm{S}p^2 = 0, \qquad \text{and} \qquad \mathrm{S}pq = 0. \tag{35}$$

Let now the four vertices of a tetrahedron be determined in this system by four quaternions p_1, p_2, p_3, p_4, which are necessarily linearly independent, since otherwise the four vertices would lie in one plane; and let q_1, q_2, q_3, q_4 be derived from them by (28) or (31). Then any two quaternions, p and q, may be expressed in the forms: —

$$\begin{cases} p = x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4, \\ q = u_1q_1 + u_2q_2 + u_3q_3 + u_4q_4; \end{cases} \tag{36}$$

in which it is unnecessary to write arbitrary factors, m and n, as in (15), since such factors are implicitly contained in the p's and q's.

Substituting these expressions in the second equation of (35), we have at once, by (28), the ordinary equation of collocation for tetrahedral coördinates:

$$u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0. \tag{37}$$

It is to be noted that the coördinates x and u, may be defined both here and in § 5, by certain anharmonic relations, and the forms p and q are therefore well adapted to the study of projective geometry. For if we assume a point p_0 and a plane q_0 such that

$$\begin{cases} p_0 = p_1 + p_2 + p_3 + p_4, \\ q_0 = q_1 + q_2 + q_3 + q_4, \end{cases} \tag{38}$$

where either the point may be assumed as any point not lying in a face of the fundamental tetrahedron, or the plane may be assumed as any plane not passing through a vertex, then for any point p,

$$x_h : x_i = (\mathrm{A}_j\mathrm{A}_k \cdot \mathrm{A}_h\mathrm{P}_0\mathrm{A}_i\mathrm{P}); \tag{39}$$

where the expression in the second member denotes the double ratio of a pencil of four planes having $A_j A_k$ for its axis, and passing respectively through A_h, P_0, A_i, P, h and i denoting any two suffixes and j and k the other two.

In like manner, if any plane q and the fixed plane q_0 intersect $A_h A_i$ in points Q_{hi} and Q'_{hi},

$$u_h : u_i = (A_i Q'_{hi} A_h Q_{hi}). \tag{40}$$

§ 9. We ought, however, to take one further step, in order to eliminate from it the metric condition that there is a relation of perpendicularity involved in the dualistic representation of a point and a plane.

Let then *any four* linearly independent quaternions be arbitrarily chosen to denote the four vertices of our tetrahedron; and let four complementary quaternions be derived from them by (28) or (31). We may now assume arbitrarily any point p_0 or else any plane q_0, taking p_0 and q_0 as in (38); and then the point or plane represented by *any* value of p or q may be determined so as to satisfy (39) or (40). The point and plane which are denoted by the same symbol must be polar to each other relatively to the surface

$$\mathrm{S}p^2 = 0; \tag{41}$$

but this may be *any* non-singular quadric; it is not in general a sphere. Thus, if we assume arbitrarily the point p_0, we obtain by (41) the plane p_0, whence we obtain the plane q_0 by the aid of (31) and (40).

In this system, as in that of § 8, it follows from the anharmonic property that the condition of the *coincidence* of two points or of two planes is that their quaternions satisfy a linear equation,

$$ap + bq = 0, \qquad \text{i. e., that} \qquad |p\bar{q}| = 0; \tag{42}$$

that the condition of the *collinearity* of three points or of three planes, is

$$ap + bq + cr = 0, \qquad \text{i. e., that} \qquad |p\bar{q}r| = 0; \tag{43}$$

and that the condition of four points belonging to a *common plane*, or four planes to a *common point*, is

$$ap + bq + cr + ds = 0, \qquad \text{i. e., that} \qquad |p\bar{q}r\bar{s}| = 0; \tag{44}$$

moreover, that the *double ratio* of four collinear points or of four collinear planes may be written

$$(pqrs) = \frac{|p\bar{q}| \times |r\bar{s}|}{|q\bar{r}| \times |s\bar{p}|}. \tag{45}$$

This last equation may be used to determine the quaternion of any point or plane, when that of an assumed point or plane is known.

§ 10. The general equation of a surface of the second order or second class in the system of § 9 may be written

$$\mathrm{S}pfp = 0\,; \tag{46}$$

where f is a linear function, which may be reduced to its self-conjugate part, while p is the quaternion of a moving point or of a moving plane. The properties of the surface may then be deduced from those of the linear function of a quaternion.

The differentiating operator ∇ may be generalized by the aid of the complete system of § 7, so as to be applicable to any function of a variable quaternion. Defining ∇ as an operator which satisfies the identity

$$\mathrm{d}_p = -\,\mathrm{S}\cdot \mathrm{d}p\nabla_p, \tag{47}$$

we may write

$$\begin{cases} p = x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4, \\ \nabla_p = q_1 \mathrm{D}_{x_1} + q_2 \mathrm{D}_{x_2} + q_3 \mathrm{D}_{x_3} + q_4 \mathrm{D}_{x_4}. \end{cases} \tag{48}$$

Thus, if $p = w + xi + yj + zk$,

$$\nabla_p = -\,\mathrm{D}_w + i\mathrm{D}_x + j\mathrm{D}_y + k\mathrm{D}_z. \tag{49}$$

It can easily be shown that the result of the application of the operator ∇_p to any function of p is independent of the arbitrary choice of the four fundamental quaternions, p_1, p_2, p_3, p_4.

It is, moreover, readily seen that the polar of any point or plane p, relatively to the quadric represented by (46), is the plane or point q, where (omitting the unnecessary factor -2)

$$q = -\tfrac{1}{2}\nabla_p \mathrm{S}pfp = \mathrm{f}p. \tag{50}$$

HARVARD UNIVERSITY,
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THREE PARTICULAR SYSTEMS OF LINES ON A SURFACE*

BY

LUTHER PFAHLER EISENHART

INTRODUCTION

CIFARELLI has established the following theorem:†

Given two quadratic differential forms

$$(a)\qquad \begin{aligned} &a_1 du^2 + 2a_2 dudv + a_3 dv^2, \\ &b_1 du^2 + 2b_2 dudv + b_3 dv^2, \end{aligned}$$

of which the first is definite, that is, $a_1 a_3 - a_2^2 > 0$; *there exists a unique real transformation of variables*

$$(b)\qquad u' = \phi(u, v), \qquad v' = \psi(u, v),$$

which changes the above forms into

$$\begin{aligned} &a_1' du'^2 + 2a_2' du'\, dv' + a_3' dv'^2, \\ &b_1' du'^2 + 2b_2' du'\, dv' + b_3' dv'^2, \end{aligned}$$

with the relations

$$\frac{a_1'}{b_1'} = \frac{a_3'}{b_3'}, \qquad a_2' = 0.$$

In order to find the transformation (b) which effects this change, one takes the Jacobian of the forms (a) and then in turn forms the Jacobian of this quadratic form and the first of (a). The resulting quadratic differential form has real linear factors; when the latter are multiplied by integrating factors and the products equated to du' and dv' respectively, the required transformation is given by quadratures.

We apply this theorem to the two fundamental forms of a surface S written as follows:

$$\text{(I)}\qquad ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

$$\text{(II)}\qquad \phi^2 = Ddu^2 + 2D'dudv + D''dv^2,$$

and to the form

$$\text{(III)}\qquad d\sigma^2 = edu^2 + 2fdu\,dv + gdv^2,$$

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† *Le Congruenze*, Annali di Matematica, ser. III, vol. 2, 1899.

which represents the square of the linear element of the spherical representation of S. Between these three quadratic forms the following linear relation holds:*

$$Kds^2 + H\phi^2 + d\sigma^2 = 0,$$

where K and H denote the total and mean curvatures respectively of the surface. The forms (I) and (III) are definite for all real surfaces, whereas the form (II) is definite only for surfaces of positive total curvature. In consequence of the above relation there are only three independent ways in which the transformation (b) can be applied to the above forms. When the surface is referred to any one of the three parametric systems so defined, the fundamental quantities for the surface satisfy the relations:

(c) $$\frac{E}{D} = \frac{G}{D''}, \qquad \frac{E}{e} = \frac{G}{g},$$

and one of three

$$1°,\ F = 0; \qquad 2°,\ f = 0; \qquad 3°,\ D' = 0.$$

The parameters corresponding to the case 3° are real only for surfaces of positive constant curvature.

We proceed in this paper to the consideration of the three particular systems of lines upon a surface which are characterized by the above relations when the lines are parametric. We consider for the lines of each system a length defined as follows: Let the surface be referred to one of the systems and at the points (u, v), $(u + du, v)$, $(u, v + dv)$ draw normals to the surface. Denote by r_u and r_v the lengths measured on the first from the point (u, v) to the points where this line is met by the lines of shortest distance to the second and the third lines respectively. We shall refer to this length as the *distance* r. It is found that the function r and the radii of normal and geodesic curvature of the directions of the lines of all three systems are simple functions of the radii of principal curvature of the surfaces. These forms are tabulated in § 3.

In § 1 the system of lines satisfying conditions (c) and 1° is studied. Their directions form angles of 45° with the tangents to the lines of curvature; on this account we call them the *mean orthogonal* lines. When S is an isothermic surface, and only in this case, the mean orthogonal lines form an isothermal system.

The system corresponding to the conditions (c) and 2° is treated in § 2 and it is found that the tangents to these curves are conjugate to the directions of the mean orthogonal lines. In the case of minimal surfaces and the sphere these curves coincide with the mean orthogonal lines. The spherical representation of these lines gives an orthogonal system and it is shown that every orthogonal system on the sphere is the image of these lines on a number of sur-

* BIANCHI, *Lezioni*, p. 116.

faces. The determination of these surfaces, when the system on the sphere is given, requires the integration of a partial differential equation of the second order and quadratures. When the system on the sphere is isothermal, the lines of curvature of the corresponding surfaces have an isothermal spherical representation.

In § 3 it is shown that the angle between directions in the system characterized by (*c*) and 3° is bisected by the directions of the lines of curvature and that no other conjugate system has this property. The existence of such a system was discovered by HOPPE* in seeking to find the conjugate system on a surface of positive curvature whose included angle is a minimum. Later PUCCI,† in considering the same problem, called these the *characteristic lines* and the included angle the *characteristic angle*. We shall make use of these terms. We find that this angle is constant only for those Weingarten surfaces for which the ratio of the principal curvature is constant. Recalling that Weingarten surfaces are characterized by the property that asymptotic lines on the sheets of the evolute correspond, we inquire whether there are any surfaces which have a similar property for characteristic lines and find that in every case the sheets are of negative curvature and consequently the characteristic lines are imaginary.

In § 4 we consider the spherical representation of characteristic lines and find the conditions which a system of lines on the sphere must satisfy in order that they may be the image of the characteristic lines. When this condition is satisfied and the coördinates of a point on the sphere are known, the further determination of the surface reduces to quadratures. As in the case of asymptotic lines the surface is unique for each system on the sphere. The expressions for the cartesian coördinates of such a surface are reduced to forms similar to the Lelieuvre‡ formulæ for surfaces whose parametric lines are asymptotic.

The paper closes with a proof of the impossibility of the successive application of the theorem of CIFARELLI for the determination of new systems of lines.

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* *Ueber das Minimum des Winkels*, etc., Grünert's Archiv, vol. 69, pp. 19–30.

† *Dell'angolo caratteristico e delle linee caratteristiche di una superficie*, Rom. Acc. L. Rend. (4), V (1889), pp. 501–7. In a paper entitled *Di alcune proprietà delle linee caratteristiche* (Rom. Acc. L. Rend. (4), V (1889), pp. 881–5) REINI showed that if normals be drawn to a surface at the points of a geodesic circle of radius ds and center (u, v), the distance between the normal at the latter point and those at the points of the circle in the direction of the characteristic lines is the maximum. RAFFY also has studied these lines in an article entitled *Sur le reseau diagonal conjugue* (Bull. de la Soc. Math. de France, vol. 30 (1902), p. 226). He set out to study the conjugate directions on a surface corresponding to equal conjugate diameters of the Dupin indicatrix of the surface. The lines satisfying this condition are the characteristic lines, as is evident from the fact that the normal radii are equal in these directions, but RAFFY does not call attention to this fact. From his definition he finds equation (24) for the definition of the directions of these lines and then he considers the forms which this equation takes when certain lines are parametric.

‡ BIANCHI, *Lezioni*, p. 130.

§ 1. *Mean orthogonal lines.*

We consider first that system of lines upon every surface which are such that when they are parametric the fundamental quantities satisfy the conditions

$$\frac{E}{D}=\frac{G}{D''}, \qquad \frac{E}{e}=\frac{G}{g}, \qquad F=0. \tag{1}$$

From the general theorem it follows that, when the surface is referred to any system of lines whatever, the directions of this special system are given by the equation

$$[E(GD-ED'')-2F(FD-ED')]du^2+[4EGD'-2F(ED''+GD)]\,dudv + [2F(GD'-FD'')-G(GD-ED'')]dv^2=0.$$

When in particular the parametric lines are the lines of curvature of the surface, the above equation reduces to

$$E_1du_1^2-G_1dv_1^2=0, \tag{2}$$

where the subscript denotes functions corresponding to the lines of curvature. If we denote by θ the angle which these lines make with the direction of the line of curvature $v=$ const., we have

$$\tan\theta=\pm 1, \tag{3}$$

that is, these curves bisect the angles between the lines of curvature. Hence we say that they form the *mean orthogonal system* on the surface.

From the general form of the equation of the directions of the lines of curvature *

$$(ED'-FD)du^2+(ED''-GD)dudv+(FD''-GD')dv^2=0, \tag{4}$$

it follows that when the surface is referred to its mean orthogonal system the lines of curvature are given by

$$Edu^2-Gdv^2=0. \tag{5}$$

Denoting by ρ_u and ρ_v the radii of normal curvature of the surface in the directions of the mean orthogonal lines $v=$ const., $u=$ const., respectively, we have from (1)

$$\rho_u=\rho_v. \tag{6}$$

By means of EULER's equation

$$\frac{1}{\rho}=\left(\frac{\cos^2\omega}{\rho_1}+\frac{\sin^2\omega}{\rho_2}\right). \tag{7}$$

* BIANCHI, l. c., p. 99.

where ρ_1 and ρ_2 are the principal radii and ω is the angle made by the section of radius ρ with the direction corresponding to ρ_1, we have

$$\frac{1}{\rho_u} = \frac{1}{\rho_v} = \tfrac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right). \tag{8}$$

Hence:

The radii of normal curvature in the directions of the mean orthogonal lines are equal to the harmonic mean of the principal radii.

From (8) it follows that

The mean orthogonal lines on a minimal surface are asymptotic;

and

The normal curvature of a surface of constant mean curvature in the direction of the mean orthogonal lines is constant all over the surface.

When in particular a mean orthogonal line on such a surface is a geodesic its first curvature is constant.

We have for any orthogonal system

$$\frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{\rho_1} + \frac{1}{\rho_2},$$

ρ, ρ' denoting the normal radii in the given orthogonal directions. We assume that $\rho_1 > \rho_2$; then since ρ_1 and ρ_2 are the limiting values we can put

$$\frac{1}{\rho} = \frac{1}{\rho_1} + h, \qquad \frac{1}{\rho'} = \frac{1}{\rho_2} - h,$$

where h is an auxiliary function satisfying the conditions

$$h \leqslant 0, \qquad h \geqslant \frac{1}{\rho_2} - \frac{1}{\rho_1}.$$

Then

$$\frac{1}{\rho\rho'} = \frac{1}{\rho_1\rho_2} + h\left(\frac{1}{\rho_2} - \frac{1}{\rho_1} - h\right).$$

From the above conditions it follows that the quantity in the parenthesis is always positive and consequently $1/\rho\rho'$ is minimum when h is zero, and maximum when h has the expression $\frac{1}{2}(1/\rho_2 - 1/\rho_1)$. Hence:

The product of the radii of normal curvature of a surface in orthogonal directions is least for the mean orthogonal lines and greatest for the lines of curvature.

From the Bertrand equation

$$\frac{1}{\tau} = \tfrac{1}{2}\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)\sin 2\theta, \tag{9}$$

where τ denotes the geodesic torsion of the curves whose tangent make an angle θ with the line of curvature corresponding to ρ_1, we find

$$\frac{1}{\tau_u} = -\frac{1}{\tau_v} = \tfrac{1}{2}\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right). \tag{10}$$

From the form of (9) it is evident that this is the maximum value for τ.*

KNOBLAUCH† shows that the *distances* r_u, r_v have the following expressions:

$$\frac{1}{r_u} = \frac{e}{D}, \qquad \frac{1}{r_v} = \frac{g}{D''}. \tag{11}$$

From the relations

$$\frac{1}{\rho_1\rho_2} = \frac{DD'' - D'^2}{EG}, \qquad \frac{1}{\rho_1} + \frac{1}{\rho_2} = -\frac{ED'' + GD}{EG}, \tag{12}$$

one has in consequence of (1)

$$\frac{1}{\rho_1} = -\frac{D}{E} + \frac{D'}{\sqrt{EG}}, \qquad \frac{1}{\rho_2} = -\frac{D}{E} - \frac{D'}{\sqrt{EG}}, \tag{13}$$

From (11), (12), (13) and (*c*) we get

$$\frac{1}{r_u} = \frac{1}{r_v} = \frac{GD^2 + ED'^2}{DEG} = \frac{\tfrac{1}{2}\left(\dfrac{1}{\rho_1^2} + \dfrac{1}{\rho_2^2}\right)}{\tfrac{1}{2}\left(\dfrac{1}{\rho_1} + \dfrac{1}{\rho_2}\right)}. \tag{14}$$

If we note that the numerator of the last member is the expression for the Casorati curvature of the surface, we have the theorem:

The distance r in the direction of a mean orthogonal line of a surface is equal to the ratio of the mean and Casorati curvatures of the surface at the point.

When the surface is minimal, r is zero, hence the lines of striction of the ruled surfaces formed by the normals to a minimal surface along its asymptotic lines are the latter lines themselves.

Let S be an isothermic surface, then if its lines of curvature are parametric, the equation of the mean orthogonal lines (2) can be brought to the form,

$$du_1^2 - dv_1^2 = 0.$$

Consequently, if we put

$$u_1 - v_1 = 2u, \qquad u_1 + v_1 = 2v,$$

u and v are the parameters of the mean orthogonal lines. Then the directions of the lines of curvature are given by the equation

$$du^2 - dv^2 = 0,$$

* BIANCHI, l. c., p. 161.

† *Einleitung in die Allgemeine Theorie der Krummen Flächen*, p. 69.

when the surface is referred to the mean orthogonal system. Comparing this with (5) we have $E = G$; in a similar manner it can be shown that when E and G are equal, E_1 and G_1 also are equal. Hence:

When a surface is isothermic, its mean orthogonal lines form an isothermal system and only in this case.

§ 2. *Lines conjugate to the mean orthogonal lines.*

The second system of lines is represented on the sphere by an orthogonal system, and when the surface is referred to this system, its coefficients satisfy the equations

$$\frac{e}{E} = \frac{g}{G}, \qquad \frac{e}{D} = \frac{g}{D''}, \qquad f = 0. \tag{15}$$

When the system of parametric lines on the surface is composed of the lines of curvature, the equation determining the directions of the system under consideration is

$$e_1 du_1^2 - g_1 dv_1^2 = 0, \tag{16}$$

from which it is seen that these lines form equal angles with the lines of curvature. The above equation can be written

$$G_1 D_1^2 du_1^2 - E_1 D_1''^2 dv_1^2 = 0,$$

from which one finds that the directions of these lines make an angle θ with the directions of the lines of curvature $v = \text{const.}$ such that

$$\tan\theta = \pm\frac{\rho_2}{\rho_1}, \tag{17}$$

where ρ_1 and ρ_2 are the principal radii. From this and (3) it follows that in the case of minimal surfaces and the sphere these lines coincide with the mean orthogonal lines; in the former case they are the asymptotic lines and in the latter any orthogonal system.

If θ and θ' denote the angles which two conjugate directions upon a surface make with one of the lines of curvature, these functions satisfy the condition

$$\tan\theta \tan\theta' = -\frac{\rho_2}{\rho_1}.$$

Comparing this result with (3) and (17), we note that the system of lines under consideration is composed of the lines conjugate to the lines of the mean orthogonal system. We shall refer to them as the *conjugates of the mean orthogonal lines.*

When this system of lines is parametric, equation (4) of the lines of curvature reduces to

$$(ED' - FD)du^2 - (GD' - FD'')dv^2 = 0,$$

which in consequence of (15) reduces to

$$edu^2 - gdv^2 = 0.$$

In a manner similar to that followed in the case of mean orthogonal lines we can establish the following theorem:

When the lines of curvature of a surface have an isothermal spherical representation, the representation of the conjugates of the mean orthogonal lines is isothermal and only in this case.

Given an orthogonal system on the sphere and put

$$t = \frac{D}{e} = \frac{D''}{g}; \tag{18}$$

the Codazzi equations* can be written

$$\begin{aligned} g\frac{\partial t}{\partial u} - \frac{\partial D'}{\partial v} + \frac{\partial}{\partial v}\log\sqrt{\frac{g}{e}}\,D' = 0, \\ e\frac{\partial t}{\partial v} - \frac{\partial D'}{\partial u} + \frac{\partial}{\partial u}\log\sqrt{\frac{e}{g}}\,D' = 0. \end{aligned} \tag{19}$$

The elimination of t leads to the following equation which D' must satisfy:

$$\begin{aligned} &\frac{1}{e}\frac{\partial^2\phi}{\partial u} - \frac{1}{g}\frac{\partial^2\phi}{\partial v^2} + \frac{1}{e}\left(\frac{\partial \log\sqrt{g}}{\partial u} - 3\frac{\partial\log\sqrt{e}}{\partial u}\right)\frac{\partial\phi}{\partial u} \\ &-\frac{1}{g}\left(\frac{\partial\log\sqrt{e}}{\partial v} - 3\frac{\partial\log\sqrt{g}}{\partial v}\right)\frac{\partial\phi}{\partial v} + \left[\frac{1}{e}\left(\frac{\partial^2}{\partial u^2}\log\sqrt{\frac{g}{e}} - \frac{\partial}{\partial u}\log\sqrt{\frac{g}{e}}\frac{\partial\log e}{\partial u}\right)\right. \\ &\left. -\frac{1}{g}\left(\frac{\partial^2}{\partial v^2}\log\sqrt{\frac{e}{g}} - \frac{\partial}{\partial v}\log\sqrt{\frac{e}{g}}\frac{\partial\log g}{\partial v}\right)\right]\phi = 0. \end{aligned} \tag{20}$$

From this it follows that any orthogonal system on the sphere is the representation of the conjugates of the mean orthogonal lines on certain surfaces. For when a particular integral D' is known of the corresponding equation (20), one gets t from (19) by a quadrature and then D and D'' from (18). The cartesian coördinates of the surface are given by quadratures from

$$\frac{\partial x}{\partial u} = -\frac{D}{e}\frac{\partial X}{\partial u} - \frac{D'}{g}\frac{\partial X}{\partial v}, \qquad \frac{\partial x}{\partial v} = -\frac{D'}{e}\frac{\partial X}{\partial u} - \frac{D''}{g}\frac{\partial X}{\partial v},$$

and similarly for y and z, where X, Y, Z are the coördinates of the sphere.

When the system upon the sphere is isothermal and the parameters are chosen so that

$$e = g = \lambda,$$

* BIANCHI, l. c., p. 120.

the equation (20) reduces to

$$\frac{\partial}{\partial u}\left(\frac{1}{\lambda}\frac{\partial \phi}{\partial u}\right)=\frac{\partial}{\partial v}\left(\frac{1}{\lambda}\frac{\partial \phi}{\partial v}\right). \tag{21}$$

If we denote by α and β the parameters of the lines of curvature of a surface with this representation of these lines, we have from the preceding discussion

$$\alpha = u + v, \qquad \beta = u - v,$$

so that the above equation becomes by change of variables

$$\frac{\partial^2 \phi}{\partial \alpha \partial \beta} - \frac{\partial \log \sqrt{\lambda}}{\partial \beta}\frac{\partial \phi}{\partial \alpha} - \frac{\partial \log \sqrt{\lambda}}{\partial \alpha}\frac{\partial \phi}{\partial \beta} = 0. \tag{22}$$

Hence the problem of finding surfaces with an isothermal spherical representation of the conjugates of the mean orthogonal lines, or, what is the same thing, such a representation of the lines of curvature reduces to the integration of an equation of Laplace with equal invariants.

In closing this discussion of the conjugates of the mean orthogonal lines upon a surface, we remark from (15) that the normals to the surface along these lines form the mean ruled surfaces of the congruence of normals.*

§ 3. *Characteristic lines.*

This system of lines is conjugate in such a way that when it is parametric its functions satisfy the conditions

$$\frac{D}{E}=\frac{D''}{G}, \qquad \frac{D}{e}=\frac{D''}{g}, \qquad D' = 0. \tag{23}$$

As we shall consider only real lines it is assumed that S has positive total curvature. The general equation of the directions of this system is readily found to be

$$\begin{aligned}&[D(GD-ED'')-2D'(FD-ED')]du^2\\ &+[2D'(GD+ED'')-4FDD'']\,dudv\\ &+[2D'(GD'-FD'')-D''(GD-ED'')]dv^2=0.\end{aligned} \tag{24}$$

When the lines of curvature are parametric, the equation for the determination of these lines is

$$D_1 du_1^2 - D_1'' dv_1^2 = 0. \tag{24'}$$

And from the general equation (4) of the lines of curvature, it follows that

*CIFARELLI, l. c.

when the surface is referred to this conjugate system, the directions of the lines of curvature are given by

$$Ddu^2 - D''dv^2 = 0. \tag{25}$$

From equations (24′) and (25) we get the theorem:

The necessary and sufficient condition that the characteristic lines upon a surface form an isothermal-conjugate system is that the lines of curvature are isothermal-conjugate.

From (24′) it is seen that the angle between the characteristic directions is bisected by the lines of curvature. Moreover, this is the only conjugate system whose included angle is thus bisected. For, the directions bisecting the angles between any parametric system on a surface are given by

$$Edu^2 - Gdv^2 = 0. \tag{26}$$

When the surface is referred to any conjugate system the equation (4) of the lines of curvature becomes

$$FDdu^2 + (GD - ED'')dudv - FD''dv^2 = 0; \tag{27}$$

and the only way in which this equation can take the above form is given by

$$GD - ED'' = 0, \tag{28}$$

which characterizes the conjugate system under discussion. Consequently this conjugate system on surfaces of positive curvature is an analogue of the asymptotic system on surfaces of negative curvature.

Denoting by ϕ the angle which the characteristic directions form with the tangent to the line of curvature of normal radius ρ_1, we have from (24′)

$$\tan\phi = \pm\sqrt{\frac{G}{E}}\frac{dv}{du} = \pm\sqrt{\frac{\rho_2}{\rho_1}}\,^{*} \tag{29}$$

Comparing (3), (17) and (29) and assuming that ρ_1 is greater than ρ_2, we see that the mean orthogonal lines make angles $\pi/4$ with the tangent to the line of curvature of normal radius ρ_1, the characteristic directions make equal and less angles, and the conjugates of the first make angles which are still less and equal to one another.

By following the method employed in the study of the mean orthogonal lines, one finds readily the expressions for the normal curvature, the geodesic torsion and the distance r for the directions of the second system of lines and the characteristic directions in terms of the principal radii. We give the results for all three systems in the following table:

* If we denote by θ the similar angle for asymptotic lines on a surface of negative curvature, we find $\tan\theta = \pm\sqrt{-\rho_2/\rho_1}$.

(30)

		I	II	III
$\frac{1}{\rho_u} = \frac{1}{\rho_v}$	$=$	$\frac{1}{2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)$	$\frac{\rho_1+\rho_2}{\rho_1^2+\rho_2^2}$	$\frac{2}{\rho_1+\rho_2}$
$\frac{1}{\tau_u} = -\frac{1}{\tau_v}$	$=$	$\frac{1}{2}\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)$	$\frac{\rho_1-\rho_2}{\rho_1^2+\rho_2^2}$	$\sqrt{\rho_1\rho_2}\,\frac{\rho_1-\rho_2}{\rho_1+\rho_2}$
$\frac{1}{r_u} = \frac{1}{r_v}$	$=$	$\frac{(\rho_1^2+\rho_2^2)}{\rho_1\rho_2(\rho_1+\rho_2)}$	$\frac{2}{\rho_1+\rho_2}$	$\frac{1}{2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)$

Between the normal radii of any two conjugate directions the following relation obtains,*

$$\rho + \rho' = \rho_1 + \rho_2.$$

By considerations similar to those used in the case of mean orthogonal lines we deduce from this relation the theorem:

The product of the radii of curvature in conjugate directions is maximum when the directions are characteristic and minimum for the lines of curvature.

From the value of r for characteristic lines, we have the theorem:

In a ruled surface formed by the normals to a surface of constant mean curvature with positive total curvature along a characteristic line the line of striction is at a constant distance from the surface.

Let S be a surface of constant mean curvature $1/2a$, then the lines of striction of all the ruled surfaces above considered lie on the surface S_0 defined by†

$$x_0 = x + 2aX, \qquad y_0 = y + 2aY, \qquad z_0 = z + 2aZ,$$

where X, Y, Z are the direction-cosines of the normal to S. Since a is constant, S_0 is parallel to S.

Between the radii of principal curvature of S and S_0 the following relations hold:‡

$$\rho_1^0 = \rho_1 - 2a, \qquad \rho_2^0 = \rho_2 - 2a,$$

from which one finds that S_0 has mean curvature $-1/2a$. From the general relations

$$D_0 = 2E\frac{a}{\rho_1\rho_2} + D\left[1 - 2a\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\right],$$

$$D_0'' = 2\frac{Ga}{\rho_1\rho_2} + D''\left[1 - 2a\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\right],$$

it results by means of (30) that D_0 and D_0'' are zero, so that the asymptotic lines on S_0 correspond to the characteristic lines on S. We have then the theorem:

* KNOBLAUCH, l. c., p. 56.
† KNOBLAUCH, l. c., p. 69.
‡ Ibid., p. 236.

The ruled surfaces formed by the normals to a surface, of positive curvature and with mean curvature $1/2a$, *along the characteristic lines cut the parallel surface of mean curvature* $-1/2a$ *in its asymptotic lines and have the latter for their lines of striction.*

From the equation (29) it is seen that on the Weingarten surfaces whose radii are bound by a relation of the form

$$\frac{\rho_1}{\rho_2} = \text{constant}, \tag{31}$$

the characteristic lines cut at constant angle and only in this case.* Moreover, the conjugates of the mean orthogonal lines also cut under constant angle, so that the configuration of all three systems is the same at all points of such a surface.

One of the characteristic properties of Weingarten surfaces is that the asymptotic lines on the sheets of the evolute correspond. We inquire whether there are any surfaces for which the characteristic lines on the sheets of the evolute correspond. Let the focal nappes be referred to the conjugate system corresponding to the lines of curvature of the surface; then the direction of the characteristic lines on the first nappe are given by the equation

$$D_1(G_1D_1 - E_1D_1')du^2 - 4F_1D_1D_1'du\,dv + D_1'(E_1D_1' - G_1D_1)dv^2 = 0,$$

where the subscripts denote that the functions belong to the first nappe; and the directions of the lines of curvature are given by

$$F_1D_1du^2 + (G_1D_1 - E_1D_1')du\,dv - F_1D_1'dv^2 = 0.$$

To get the equations for the second nappe we have only to change the subscript. Now the necessary and sufficient condition that the characteristic lines correspond on the two nappes is

$$\frac{D_1(G_1D_1 - E_1D_1')}{D_2(G_2D_2 - E_2D_2')} = \frac{F_1D_1D_1'}{F_2D_2D_2'} = \frac{D_1'(E_1D_1' - G_1D_1)}{D_2'(E_2D_2' - G_2D_2)},$$

which reduces to

$$\frac{G_1D_1 - E_1D_1'}{G_2D_2 - E_2D_2'} = \frac{F_1D_1}{F_2D_2} = \frac{F_1D_1'}{F_2D_2'}.$$

From the last two we have that the asymptotic lines correspond on the two sheets and consequently S is a Weingarten surface. Moreover, from the above equation of the lines of curvature it follows that the lines of curvature on the two nappes correspond and conversely when this condition is satisfied the above rela-

* It is evident that on the surfaces whose radii satisfy (31) any function of the tangent of the angle between the characteristic lines is constant. MASSIMI (Atti dell' Accad. Pont. dei Nuovi Lincei, vol. 52, p. 130) has shown that this is true for every conjugate system of lines on these Weingarten surfaces and only for these.

tions hold. But the only W surfaces whose focal sheets have corresponding lines of curvature are those corresponding to a relation of the form

$$\rho_1 - \rho_2 = \text{constant.}^*$$

But the focal sheets in this case are pseudo-spherical surfaces and consequently the characteristic lines are imaginary.

In the preceding discussion we have not treated the case where the conjugate lines upon the sheets corresponding to the lines of curvature on the surface are the characteristic lines. Making use of the expression given by KNOBLAUCH † for the fundamental coefficients of the two nappes in terms of the functions for S referred to its lines of curvature, we have the following conditions to be satisfied

$$\left[G\rho_1^2 \frac{\partial \rho_1}{\partial u}\frac{\partial \rho_2}{\partial u} - E\rho_2^2\left(\frac{\partial \rho_1}{\partial v}\right)^2 - EG(\rho_1-\rho_2)^2 \right]\frac{\partial \rho_1}{\partial u} = 0,$$

$$\left[E\rho_2^2 \frac{\partial \rho_1}{\partial v}\frac{\partial \rho_2}{\partial v} - G\rho_1^2\left(\frac{\partial \rho_2}{\partial u}\right)^2 - EG(\rho_1-\rho_2)^2 \right]\frac{\partial \rho_2}{\partial v} = 0.$$

If ρ_1 is a function of v alone, or ρ_2 of u alone, one of the focal nappes reduces to a curve, so that this case cannot arise. Consequently the parentheses must be zero, but these are proportional † to the mean curvatures of the focal sheets, so that the latter must be minimal surfaces and therefore surfaces of negative curvature. Hence the theorem:

There are no surfaces whose focal sheets are of positive total curvature and have their characteristic lines in correspondence.

§ 4. *Spherical representation of characteristic lines.*

Consider a surface referred to its real characteristic lines. The following relations hold between the differential quotients of its rectangular coördinates and of the direction-cosines of the normal ‡

$$\frac{\partial x}{\partial u} = -\frac{D}{eg-f^2}\left(g\frac{\partial X}{\partial u} - f\frac{\partial X}{\partial v}\right),$$

$$\frac{\partial x}{\partial v} = \frac{D''}{eg-f^2}\left(f\frac{\partial X}{\partial u} - e\frac{\partial X}{\partial v}\right),$$

with analogous expressions in y and z. If we put

$$\frac{D\sqrt{g}}{\sqrt{e}\cdot\sqrt{eg-f^2}} = \frac{D''\sqrt{e}}{\sqrt{g}\cdot\sqrt{eg-f^2}} = \lambda, \tag{32}$$

* BIANCHI, l. c., p. 234.

† L. c., p. 218.

‡ BIANCHI, l. c., p. 119.

the second of conditions (23) is satisfied and

$$\lambda^2 = \frac{1}{K}. \tag{33}$$

When the substitution is made in the above, they become

$$\frac{\partial x}{\partial u} = -\lambda \sqrt{\frac{e}{g}} \left(\frac{g}{\sqrt{eg - f^2}} \frac{\partial X}{\partial u} - \frac{f}{\sqrt{eg - f^2}} \frac{\partial X}{\partial v} \right),$$

$$\frac{\partial x}{\partial v} = \lambda \sqrt{\frac{g}{e}} \left(\frac{f}{\sqrt{eg - f^2}} \frac{\partial X}{\partial u} - \frac{e}{\sqrt{eg - f^2}} \frac{\partial X}{\partial v} \right),$$

which by means of a set of formulæ given by BIANCHI* can be brought to the form

$$\frac{\partial x}{\partial u} = \lambda \sqrt{\frac{e}{g}} \begin{vmatrix} X & Z \\ \frac{\partial Y}{\partial v} & \frac{\partial Z}{\partial v} \end{vmatrix}, \qquad \frac{\partial x}{\partial v} = -\lambda \sqrt{\frac{g}{e}} \begin{vmatrix} Y & Z \\ \frac{\partial Y}{\partial u} & \frac{\partial Z}{\partial u} \end{vmatrix}. \tag{34}$$

By means of (32) the Codazzi equations can be written

$$\begin{aligned} \frac{\partial \log \lambda}{\partial u} &= \frac{\partial}{\partial u} \log \sqrt{\frac{e}{g}} - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}' \frac{e}{g} - \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}', \\ \frac{\partial \log \lambda}{\partial v} &= \frac{\partial}{\partial v} \log \sqrt{\frac{g}{e}} - \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}' \frac{g}{e} - \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}', \end{aligned} \tag{35}$$

where the Christoffel symbols $\{ {rs \atop t} \}'$ are formed with respect to the linear element of the spherical representation. Expressing the condition of integrability of the above equations, we are brought to the theorem:

The necessary and sufficient condition that a system of lines upon the sphere shall be the spherical representation of the characteristic lines upon a surface is that the following equation be satisfied:

$$\frac{\partial}{\partial u} \left[\begin{Bmatrix} 11 \\ 2 \end{Bmatrix}' \frac{g}{e} + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}' \right] + \frac{\partial}{\partial v} \log \sqrt{\frac{e}{g}} = \frac{\partial}{\partial v} \left[\begin{Bmatrix} 22 \\ 1 \end{Bmatrix}' \frac{e}{g} + \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}' \right] + \frac{\partial}{\partial u} \sqrt{\frac{g}{e}}. \dagger \tag{36}$$

Where this condition is satisfied and the coördinates of the sphere have been found, the complete determination of the surface is given by the quadratures (35) and (34).

When a surface is referred to any conjugate system, one has ‡

$$E = \frac{gD^2}{eg - f^2}, \qquad F = -\frac{fDD''}{eg - f^2}, \qquad G = \frac{eD''^2}{eg - f^2}. \tag{37}$$

* L. c., p. 128.

† This equation of condition has been found by RAFFY, l. c., p. 232.

‡ BIANCHI, l. c., p. 117.

When the system is composed of the characteristic lines, these relations can, by means of (32), be put in the form

$$E = \lambda^2 e, \qquad F = -\lambda^2 f, \qquad G = \lambda^2 g. \tag{38}$$

Conversely, in order that equations (37) can take this form, it is necessary that

$$\frac{gD^2}{e} = DD'' = \frac{eD''^2}{g},$$

or

$$\frac{D}{e} = \frac{D''}{g}.$$

Hence the theorem:

The necessary and sufficient condition that the linear element of a surface of positive curvature, referred to a conjugate system, can be written in the form

$$ds^2 = \lambda^2(edu^2 - 2fdu\,dv + gdv^2) \tag{39}$$

is that the conjugate system be composed of the characteristic lines.

Here again the analogy between characteristic lines and asymptotic lines is apparent.

If we put

$$\xi = \sqrt{\lambda}X, \qquad \eta = \sqrt{\lambda}Y, \qquad \zeta = \sqrt{\lambda}Z,$$

the equations (34) reduce to

$$\frac{\partial x}{\partial u} = \sqrt{\frac{e}{g}}\begin{vmatrix} \eta & \zeta \\ \frac{\partial \eta}{\partial v} & \frac{\partial \zeta}{\partial v} \end{vmatrix}, \qquad \frac{\partial x}{\partial v} = \sqrt{\frac{g}{e}}\begin{vmatrix} \eta & \zeta \\ \frac{\partial \eta}{\partial u} & \frac{\partial \zeta}{\partial u} \end{vmatrix},$$

and similar expressions in y and z. From the following two equations which X, Y, Z satisfy*

$$\frac{\partial^2\phi}{\partial u^2} - \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}'\frac{\partial\phi}{\partial v} + e\phi = 0,$$

$$\frac{\partial^2\phi}{\partial v^2} - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}'\frac{\partial\phi}{\partial v} + g\phi = 0,$$

we find by aid of (35) that they are also solutions of

$$\sqrt{\frac{g}{e}}\frac{\partial^2\phi}{\partial u} + \sqrt{\frac{e}{g}}\frac{\partial^2\phi}{\partial v^2} + \sqrt{\frac{g}{e}}\left[\frac{\partial \log\lambda}{\partial u} + \frac{\partial}{\partial u}\log\sqrt{\frac{g}{e}}\right]\frac{\partial\phi}{\partial u}$$

$$+ \sqrt{\frac{e}{g}}\left[\frac{\partial \log\lambda}{\partial v} + \frac{\partial}{\partial v}\log\sqrt{\frac{e}{g}}\right]\frac{\partial\phi}{\partial v} + 2\sqrt{eg}\phi = 0.$$

* BIANCHI, l. c., p. 119.

If now we put

$$\sqrt{\lambda}\phi = \theta,$$

the above equation reduces to

$$\frac{\partial}{\partial u}\left(\sqrt{\frac{g}{e}}\frac{\partial\theta}{\partial u}\right) + \frac{\partial}{\partial v}\left(\sqrt{\frac{e}{g}}\frac{\partial\theta}{\partial v}\right) = M\theta,$$

$$M \equiv \frac{1}{\sqrt{\lambda}}\left[\frac{\partial}{\partial u}\left(\sqrt{\frac{g}{e}}\frac{\partial\sqrt{\lambda}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\sqrt{\frac{e}{g}}\frac{\partial\sqrt{\lambda}}{\partial v}\right)\right] - 2\sqrt{eg}.$$

It is seen that this equation is satisfied by ξ, η, ζ.

Conversely, given any equation of the form

$$\frac{\partial}{\partial u}\left(\frac{1}{\mu}\frac{\partial\theta}{\partial u}\right) + \frac{\partial}{\partial v}\left(\mu\frac{\partial\theta}{\partial v}\right) = M\theta,$$

where μ and M are any functions of u and v, the surface whose coördinates are obtained by quadratures from

$$x = \int \mu \begin{vmatrix} \eta & \zeta \\ \dfrac{\partial\eta}{\partial v} & \dfrac{\partial\zeta}{\partial v} \end{vmatrix} du - \frac{1}{\mu}\begin{vmatrix} \eta & \zeta \\ \dfrac{\partial\eta}{\partial u} & \dfrac{\partial\zeta}{\partial u} \end{vmatrix} dv,$$

and similarly for y and z, where ξ, η, ζ are solutions of the above equation, has the curves $u = \text{const.}$, $v = \text{const.}$, for a conjugate system. Moreover, if the corresponding functions e and g satisfy the condition

$$\frac{e}{g} = \mu^2,$$

these curves are the characteristic lines. From (39) it follows that a necessary condition that these lines be characteristic on the above surface is

$$\frac{E}{G} = \mu^2,$$

but from (37) it is evident that this is not a sufficient condition.

We have seen that the characteristic lines from an isothermal conjugate system only on surfaces with isothermal conjugate lines of curvature. In this case by a suitable choice of the parameters of the characteristic lines we have D and D'' equal, so that e and g are equal. When e and g are made equal in the preceding equations, they take the forms given by BIANCHI to the equations of surfaces referred to any isothermal conjugate system.

In closing we call attention to the fact that the differential equations of the three systems of lines which have been considered reduces, when the surface is referred to its lines of curvature, to the form

$$\psi^2 \equiv \lambda du^2 - \mu dv^2 = 0,$$

when λ, μ have the respective values E, G; e, g; D, D'' according as this equation defines curves of the first, second or third class. For this parametric system the forms ds^2, ϕ^2 and $d\sigma^2$ lack the middle term. The Jacobian of ψ^2 and any of the last three is of the form

$$\chi^2 \equiv t\,du dv,$$

where t vanishes in case S is a minimal surface, which class of surfaces must be excluded from the consideration as we have seen before. If in accordance with the CIFARELLI theorem we form the Jacobian of χ^2 and ds^2, ϕ^2 or $d\sigma^2$, we are brought to one of the forms ψ^2. The only one of the forms ψ^2 which is definite is

$$Ddu^2 - D''dv^2,$$

for surfaces of negative curvature. The Jacobian of this and χ^2 gives ϕ^2, hence we are brought to asymptotic lines. Since the latter may be considered as forming a system of lines from which we started, we may say that *the repeated application of this theorem of* CIFARELLI *does not give rise to any lines other than those of the three systems considered.*

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ON RULED SURFACES WHOSE FLECNODE CURVE INTERSECTS EVERY GENERATOR IN TWO COINCIDENT POINTS*

BY

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The formulæ and the theorems developed in my recent paper *Studies in the general theory of ruled surfaces* are not directly applicable to the case when $\theta_4 = 0$, i. e., when the flecnode curve intersects every generator in two coincident points. The general notions, employed in that paper,‡ may however be applied to this special case as well, and give rise to a number of interesting and important considerations.

§ 1. *The covariant* C_3.

The interpretation of the covariant C_3 given in *Covariants* and *Studies* is complete for the case $\theta_4 \neq 0$. It breaks down absolutely for $\theta_4 = 0$.

This covariant is

$$C_3 = \alpha z - \beta y, \tag{1}$$

where

$$\begin{aligned} \alpha &= 2(u_{11} - u_{22})\rho + 4u_{12}\sigma + \tfrac{1}{2}(v_{11} - v_{22})y + v_{12}z, \\ \beta &= 4u_{21}\rho - 2(u_{11} - u_{22})\sigma + v_{21}y - \tfrac{1}{2}(v_{11} - v_{22})z. \S \end{aligned} \tag{2}$$

We have

$$\theta_4 = (u_{11} - u_{22})^2 + 4u_{12}u_{21} = 0.$$

Let us assume that the curve C_y is the flecnode curve, so that $u_{12} = 0$. We shall then have also $u_{11} - u_{22} = 0$. Further we may assume $p_{11} = p_{22} = 0$.

We have therefore

$$u_{12} = u_{11} - u_{22} = 0, \qquad p_{11} = p_{22} = 0,$$

whence

$$\alpha = p_{12}u_{21}y, \qquad \beta = 4u_{21}\rho + v_{21}y - p_{12}u_{21}z. \tag{3}$$

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† Of the Carnegie Institution of Washington.

‡ Hereafter referred to as *Studies*.

§ *Covariants*, p. 448.

If a transformation of the independent variable be made by putting $\xi = \xi_{(x)}$, we find that for the new system of differential equations

$$\bar{v}_{21} = \frac{1}{(\xi')^3}(v_{21} - 4u_{21}\eta), \qquad \eta = \frac{\xi''}{\xi'}.^*$$

Therefore if $u_{21} \neq 0$, i. e., if S is not a quadric, we can always choose η in just one way so as to make $\bar{v}_{21} = 0$. We obtain therefore a perfectly definite surface of the congruence Γ, which we will call its *principal* surface, and which we shall characterize geometrically farther on.

Let us assume that the variable x has already been chosen in such a way that S', the derivative of S with respect to x, shall coincide with the principal surface of Γ. Then $v_{21} = 0$, and (3) becomes

$$\alpha = p_{12}u_{21}y, \qquad \beta = 4u_{21}\rho - p_{12}u_{21}z = u_{21}\bar{\beta},$$

where

$$\bar{\beta} = 4\rho - p_{12}z.$$

We have further in general (*Covariants*, p. 430),

$$\rho = 2y' + p_{11}y + p_{12}z,$$

whence

$$2y' = \rho - p_{12}z.$$

The point, whose coördinates are $y_1', \cdots, y_4'$, is therefore obviously the intersection of the tangent to the flecnode curve with the line $P_z P_\rho$. The point P_β whose coördinates are given by $\beta_1 \cdots \beta_4$ is also on the line $P_z P_\rho$, and the cross ratio of the four points $P_\rho P_\beta P_z P_{y'}$ is $\frac{1}{4}$. The point P_α obviously coincides with P_y.

The ruled surface which the covariant C_3 adjoins to S may therefore be defined as follows. In the plane tangent to S at its flecnode P_y, construct a line passing through P_y such that it, together with the generator, the flecnode tangent and the tangent of the flecnode curve shall constitute a plane pencil whose anharmonic ratio is $\frac{1}{4}$. The locus of these lines is the required ruled surface. Moreover the points of any generator of this surface are, by means of the covariant C_3 put into a one-to-one correspondence with those of y. The lines joining corresponding points pass through P_ρ, that point of the principal surface of the congruence Γ which corresponds to P_y.

It only remains to give a characteristic geometric property of the principal surface of Γ. For this purpose let us assume, in addition to our previous hypotheses, that C_z is an asymptotic curve on S, i. e., let $p_{21} = 0$. Then (*Congruence*, p. 188).

* *Congruence*, p. 187.

$$2\sigma' = u_{21}y + u_{11}z,$$

(4)

$$2\sigma'' = u'_{21}y + (u'_{11} - \tfrac{1}{2}u_{21}p_{12})z + \tfrac{1}{2}(u_{21}\rho + u_{11}\sigma),$$

and

$$v_{21} = 2u'_{21}.$$

The first equation shows that the tangent to the curve C_σ at P_σ intersects the generator g of S. Denote this point of intersection by $P_{\sigma'}$. The locus of the point $P_{\sigma'}$ is therefore a curve on S, $C_{\sigma'}$. Its tangent at $P_{\sigma'}$ is obtained by joining $P_{\sigma'}$ to the point $P_{\sigma''}$ defined by the second equation (4). But $P_{\sigma''}$ is in the plane $P_z P_\rho P_\sigma$ if and only if $v_{21} = 0$. Put

$$\tau = u_{21}\rho + u_{11}\sigma,$$

then P_τ is a point on $P_\rho P_\sigma$ such that the line joining it to $P_{\sigma'}$ is a generator of the hyperboloid H osculating S along g. We see that the tangent to C_σ, intersects $P_z P_\tau$ if and only if $v_{21} = 0$.

The principal surface of the congruence Γ' is therefore defined by the following theorem:

We draw upon the ruled surface S any asymptotic line C_z, and upon any surface S' of the congruence Γ the curve C_σ which corresponds to it, P_z and P_σ being corresponding points of the two curves. The tangent to C_σ at P_σ always intersects the generator g of S, which passes through P_z, in a certain point $P_{\sigma'}$, whose locus gives a curve $C_{\sigma'}$ upon S. Upon S' a point P_τ is constructed such that the line $P_\tau P_{\sigma'}$ shall be a generator of the hyperboloid osculating S along g. The tangent to $C_{\sigma'}$ at $P_{\sigma'}$ intersects the line $P_z P_\tau$ if and only if the surface S' is the principal surface of the congruence.

§ 2. *The derivative conic.*

In the general case where $\theta_4 \neq 0$, the hyperboloid H osculating S along g and the hyperboloid H' osculationg S' along g' intersect along g' and a space cubic which we have called the derivative cubic. Moreover this cubic does not degenerate unless either S has a straight line directrix or S' is developable.

In the present case however, H and H' have besides g' the straight line f in common, i. e., the flecnode tangent passing through P_y. The rest of their intersection is therefore a conic which we shall call the *derivative conic*.

We proceed to prove these statements and to derive the equations of the derivative conic. Taking as tetrahedron of reference the tetrahedron $P_y P_z P_\rho P_\sigma$, the equation of H is

$$x_1 x_4 - x_2 x_3 = 0,$$

and that of H'

$$(u_{22}x_1 - u_{21}x_2)[Jx_4 - (\lambda_{12}u_{22} - \lambda_{22}u_{12})x_1 + (\lambda_{12}u_{21} - \lambda_{22}u_{11})x_2]$$
$$+ (u_{12}x_1 - u_{11}x_2)[Jx_3 - (\lambda_{11}u_{22} - \lambda_{21}u_{12})x_1 + (\lambda_{11}u_{21} - \lambda_{21}u_{11})x_2] = 0,$$

as we have shown in *Studies*, equations (1) and (3).

In our case we may put

$$u_{12} = u_{11} - u_{22} = 0, \qquad J = u_{11}^2$$

whence

$$(5) \qquad v_{11} - v_{22} = 2p_{12}u_{21}, \qquad v_{12} = 0, \qquad v_{21} = 2u'_{21} - (p_{11} - p_{22})u_{21}$$

and

$$(6) \qquad \begin{aligned} 2J\lambda_{11} &= -u_{11}v_{11}, & 2J\lambda_{12} &= 0, \\ 2J\lambda_{21} &= u_{21}v_{22} - u_{11}v_{21}, & 2J\lambda_{22} &= -u_{11}v_{22}. \end{aligned}$$

We find, therefore, for H' the equation

$$(u_{11}x_1 - u_{21}x_2)[Jx_4 - \lambda_{22}u_{11}x_2] - u_{11}x_2[Jx_3 - \lambda_{11}u_{11}x_1 + (\lambda_{11}u_{21} - \lambda_{21}u_{11})x_2] = 0,$$

or

$$(7) \quad Ju_{11}(x_1x_4 - x_2x_3) + u_{11}^2(\lambda_{11} - \lambda_{22})x_1x_2 - Ju_{21}x_2x_4 + [u_{11}^2\lambda_{21} - u_{11}u_{21}(\lambda_{11} - \lambda_{22})]x_2^2 = 0,$$

while the equation of H is

$$x_1x_4 - x_2x_3 = 0.$$

Both equations are satisfied by $x_1 = x_2 = 0$, as well as by $x_2 = x_4 = 0$ which proves that g' and f are lines upon both of these hyperboloids. They must therefore have also a conic in common, whose plane must, according to (7), have the equation

$$u_{11}(\lambda_{11} - \lambda_{22})x_1 - u_{11}u_{21}x_4 + [u_{11}\lambda_{21} - u_{21}(\lambda_{11} - \lambda_{22})]x_2 = 0.$$

If we put for abbreviation

$$(8) \quad A = u_{11}(\lambda_{11} - \lambda_{22}), \qquad B = -u_{11}\lambda_{21} + u_{21}(\lambda_{11} - \lambda_{22}), \qquad C = u_{11}u_{21},$$

whence

$$(9) \qquad u_{11}A + p_{12}C = 0,$$

we have therefore, as the *equations of the derivative conic*

$$(10) \qquad Ax_1 - Bx_2 - Cx_4 = 0, \qquad x_1x_4 - x_2x_3 = 0.$$

We can also express the coördinates of any point on the conic in terms of a parameter t. Any point on the hyperboloid H can be represented in the form

$$x_1 = ut, \qquad x_2 = t, \qquad x_3 = u, \qquad x_4 = 1.$$

This point is, moreover, a point of the conic if the condition

$$Aut - Bt - C = 0$$

is satisfied, whence

$$u = \frac{C + Bt}{At}.$$

If we substitute into the above equations for $x_1 \cdots x_4$ and multiply by At, we find

$$(11) \qquad x_1 = t(C + Bt), \qquad x_2 = At^2, \qquad x_3 = C + Bt, \qquad x_4 = At$$

as the parametric equations of the conic, or in homogeneous form

$$(12) \qquad x_1 = t_1\psi, \qquad x_2 = At_1^2, \qquad x_3 = t_2\psi, \qquad x_4 = At_1t_2,$$

where

$$(13) \qquad \psi = Bt_1 + Ct_2.$$

The conic of course always passes through P_ρ. The first question which we naturally ask is this: when does the conic degenerate into a pair of lines? Clearly this can only happen if the plane

$$Ax_1 - Bx_2 - Cx_4 = 0$$

intersects the hyperboloid H in a pair of lines, i. e., if it is tangent to H. Moreover since this plane contains P_ρ, it must in that case contain at least one of the two generators of H which pass through P_ρ. If it contains that one which passes also through P_σ we must have $C = 0$, i. e., since $u_{21} \neq 0$ (S not being a quadric), $u_{11} = 0$. S' must therefore be a developable. If however this plane contains the other generator through P_ρ, namely that one which passes through P_y, we must have $A = 0$, which gives either $u_{11} = 0$ as before or else $p_{12} = 0$, in which case the flecnode curve C_y would be a straight line.

Therefore, the derivative conics degenerate if and only if the surface S has a straight line directrix, or else if the derivative of S with respect to x is one of the developable surfaces of the congruence Γ.

By an investigation similar to that in *Studies*, pp. 230–232, we obtain the further result:

Two consecutive derivative conics never intersect unless they degenerate.

§ 3. *The developable surface generated by the plane of the derivative conic.*

As x changes, the plane of the conic C_x envelops a developable surface, the equations of whose generator we shall now proceed to determine.

Let us form $\partial\phi/\partial x$ under the assumption that $t_1 : t_2$ is independent of x. Then

$$\phi + \frac{\partial\phi}{\partial x}\delta x$$

will represent any point on the derivative conic $C_{x+\delta x}$ belonging to the argument $x + \delta x$, where δx is an infinitesimal. The plane of this conic will be determined by any three points upon it. We have (again assuming $p_{21} = 0$),

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} = {} & y\left[t_1(B't_1 + C't_2) + \tfrac{1}{2}(Bt_1 + Ct_2)u_{11}t_2 + \tfrac{1}{2}At_1t_2u_{21}\right] \\
& + z\left[A't_1^2 - \tfrac{1}{2}(Bt_1 + Ct_2)p_{12}t_1 + \tfrac{1}{2}At_1t_2u_{11}\right] \\
& + \rho\left[t_2(B't_1 + C't_2) + \tfrac{1}{2}(Bt_1 + Ct_2)t_1\right] \\
& + \sigma\left[A't_1t_2 - \tfrac{1}{2}(Bt_1 + Ct_2)p_{12}t_2 + \tfrac{1}{2}At_1^2\right].
\end{aligned} \tag{14}
$$

We can obtain three points of the conic $C_{x+\delta x}$ by putting $t_1 = 0, t_2 = 1$; $t_1 = 1, t_2 = 0$; $t_1 = +C, t_2 = -B$, which last set of values corresponds to $\psi = 0$. Therefore the equation of the plane of $C_{x+\delta x}$ is

$$
\begin{vmatrix}
x_1 & x_2 & x_3 & x_4 \\
Cu_{11}\delta x & 0 & 2C+2C'\delta x & -Cp_{12}\delta x \\
2B+2B'\delta x & 2A+(2A'-Bp_{12})\delta x & B\delta x & A\delta x \\
[2C(B'C-BC') & 2AC^2+(2A'C^2 & -2B(B'C & -2ABC+(AC^2 \\
\quad -ABCu_{21}]\delta x & \quad -ABCu_{11})\delta x & \quad -BC')\delta x & \quad -2A'BC)\delta x
\end{vmatrix} = 0.
$$

If this determinant be developed, retaining of course only the terms of the first order in δx, we find that the planes of C_x and of $C_{x+\delta x}$ intersect along the line

$$
\begin{aligned}
Ax_1 - Bx_2 - Cx_4 &= 0, \\
ADx_1 - Ex_2 - CFx_4 &= 0,
\end{aligned} \tag{15}
$$

where

$$
\begin{aligned}
D &= 4A'C + 2AC' - BCp_{12}, \\
E &= 2BCA' + 2ACB' + 2ABC' - AC^2, \\
F &= 4AC' + 2A'C - ABu_{11} + A^2u_{21} = 4AC' + 2A'C + Au_{11}^2\lambda_{21}.
\end{aligned} \tag{16}
$$

Equations (15) *are therefore the equations of the generators of the developable surface.*

This proof would not be valid if either C or B were zero. For then the third of the three points of the conic $C_{x+\delta x}$ which we have used to determine its plane would coincide with one of the other two. If $C = 0$ the conic degenerates. Since however we might in the case $B = 0$ choose three other points of $C_{x+\delta x}$, as we might also do in the general case, the result will obviously be obtained from the general case by substituting $B = 0$.

We see from (15) that the generator of the developable surface passes through P_ρ and that it intersects the plane $x_3 = 0$, in the point P_χ or

$$[C(E - BF),\ AC(D - F),\ 0,\ A(E - BD)],$$

which we may therefore represent by the expression

$$\chi = C(E - BF)y + AC(D - F)z + A(E - BD)\sigma. \tag{17}$$

Since the surface generated by this line joining P_ρ to P_x is developable, it must be possible to find four functions $\alpha, \beta, \gamma, \delta$ of x such that

$$\alpha\rho + \beta\chi + \gamma\rho' + \delta\chi' = 0. \tag{18}$$

Now we have

$$\rho' = \tfrac{1}{2}(u_{11}y - p_{12}\sigma),$$

and we find

$$\chi' = Gy + Hz + M\rho + N\sigma, \tag{19}$$

where

$$\begin{aligned} G &= C(E' - BF' - B'F) + C'(E - BF) + \tfrac{1}{2}u_{21}A(E - BD), \\ H &= (AC' + A'C)(D - F) + AC(D' - F') - \tfrac{1}{2}p_{12}C(E - BF) \\ &\qquad + \tfrac{1}{2}u_{11}A(E - BD), \\ M &= \tfrac{1}{2}C(E - BF), \\ N &= A'(E - BD) + A(E' - BD' - B'D) + \tfrac{1}{2}AC(D - F). \end{aligned} \tag{20}$$

If we substitute these values of χ' and ρ' and also the expression (17) for χ into (18), we find that $\alpha, \beta, \gamma, \delta$ must satisfy the equations

$$\begin{aligned} C(E - BF)\beta + \tfrac{1}{2}u_{11}\gamma + G\delta &= 0, \\ AC(D - F)\beta \qquad\quad + H\delta &= 0, \\ A(E - BD)\beta - \tfrac{1}{2}p_{12}\gamma + N\delta &= 0, \\ \alpha + M\delta &= 0. \end{aligned} \tag{21}$$

Therefore the determinant of the first three equations, which expanded becomes

$$\tfrac{1}{2}p_{12}C(D - F)[NC - GA + HB], \tag{22}$$

must vanish identically; i. e., since the other factors do not vanish identically, we must have

$$NC - GA + HB = 0. \tag{23}$$

We may also verify (23) directly. For we find from (20),

$$\begin{aligned} NC - GA + HB &= F(ACB' - \tfrac{1}{2}AC^2 - BA'C + \tfrac{1}{2}B^2Cp_{12}) \\ &\quad - D(ACB' - \tfrac{1}{2}AC^2 - BAC' - \tfrac{1}{2}A^2Bu_{21} + \tfrac{1}{2}AB^2u_{11}) \\ &\quad + E(A'C - AC' - \tfrac{1}{2}A^2u_{21} - \tfrac{1}{2}CBp_{12} + \tfrac{1}{2}ABu_{11}) \\ &= \tfrac{1}{2}[F(E - BD) - D(E - BF) + E(D - F)] = 0. \end{aligned}$$

We can now determine the edge of regression of the developable surface. If $\gamma\rho + \delta\chi$ is a point on this curve, its tangent constructed at that point must coincide with the generator of the developable, i. e.,

$$\gamma'\rho + \delta'\chi + \gamma\rho' + \delta\chi' = \lambda\chi + \mu\rho \tag{24}$$

or

$$(\gamma' - \mu)\rho + (\delta' - \lambda)\chi + \gamma\rho' + \delta\chi' = 0,$$

which is identical with (18) if we put there

$$\alpha = \gamma' - \mu, \qquad \beta = \delta' - \lambda.$$

But on account of (23) we can determine α, β, γ, δ so as to satisfy (18); we can therefore determine λ, μ, γ, δ so as to satisfy (24). Moreover we find

$$\gamma : \delta = -2GA(D-F) + 2H(E-BF) : u_{11}A(D-F). \tag{25}$$

Therefore the edge of regression of the developable is given by the expression

$$\begin{aligned}\kappa = {} & [-2GA(D-F) + 2H(E-BF)]\rho \\ & + u_{11}A(D-F)[C(E-BF)y + AC(D-F)z + A(E-BD)\sigma].\end{aligned} \tag{26}$$

We see from (15) that the generator of the developable surface coincides with one of the generators of H which passes through P_ρ, only if either A or C vanishes, i. e., either if S' is developable or if S has a straight line directrix, in which cases the derivative conic degenerates. *The generator of the developable is tangent to H at P_ρ, neglecting the cases just mentioned only if* $D - F = 0$. As (26) shows, *the cuspidal edge of the developable then coincides with C_ρ.* If the expressions for A, B, C be substituted into the condition $D - F = 0$, or

$$2(A'C - AC') - BCp_{12} + ABu_{11} - A^2u_{21} = 0,$$

it becomes

$$4\frac{u'_{11}}{u_{11}} - 2\frac{p'_{12}}{p_{12}} - 2\frac{u'_{21}}{u_{21}} = 0,$$

which gives on integration

$$\frac{u_{11}^4}{p_{12}^2 u_{21}^2} = \text{const.} \tag{27}$$

If $E - BF = 0$, P_κ lies in the plane $P_\rho P_z P_\sigma$, and if $E - BD = 0$ in the plane $P_y P_z P_\rho$.

It will clearly be possible to characterize special classes of ruled surfaces (for $\theta_4 = 0$) by special properties of the developable surfaces here considered. As we are here dealing with the general theory of such surfaces only, we will not go into such details at present.

We will also merely mention the fact that the relation of S to its flecnode surface F is especially close in this case ($\theta_4 = 0$). In fact S is also the flecnode surface of F. Moreover, the same hyperboloid H which osculates S along g, also osculates F along the corresponding generators f of F. The con-

gruence Γ' which belongs to the surface F in the same way as Γ does to S, is therefore made up of the generators of the second set on the osculating hyperboloids of S, those of the first set constituting the lines of the congruence Γ. All of these remarks follow easily from the equations of the flecnode surface, which, under the assumptions $p_{11} = p_{22} = p_{21} = 0$, assume the form

$$(28)\qquad \begin{aligned} &y'' - 2\frac{q_{12}}{p_{12}}y' - \rho' - q_{11}y + \frac{q_{12}}{p_{12}}\rho = 0, \\ &\rho'' + 4q_{11}y' - 2\frac{q_{12}}{p_{12}}\rho' + \left[2q_{11}' - p_{12}q_{21} - 4q_{11}\frac{q_{12}}{p_{12}}\right]y - q_{11}\rho = 0. \end{aligned}$$

Moreover, since S' is a developable surface of the congruence Γ if $q_{11} = 0$, and since (28) shows that C_ρ is then an asymptotic curve on F, we see that the developable surfaces of the congruence intersect its focal surface F along asymptotic lines, as it should according to the general theory of congruences.

PARIS, *October* 30, 1903.

ON THE AUTOMORPHIC FUNCTIONS OF THE GROUP (0, 3; 2, 6, 6)*

BY

J. I. HUTCHINSON

In the recently published work of FRICKE and KLEIN on automorphic functions,† attention is called to the desirability of studying groups, which, like the modular group, arise from the monodromy of the branch points of a given Riemann surface. On this suggestion I have examined a number of such cases, and present in the following pages some details relative to one of the simplest of these.

In § 1 the group arising from the monodromy of the branch-points of the given surface is determined and is found to be the group (0, 3; 2, 6, 6).‡ In § 2 the "multiplicative" functions are considered, these being slightly more general than the POINCARÉ theta functions and including the latter as a subclass. Some general properties of such functions are discussed, and the simplest functions for the present group are determined. In § 3 definite analytic expressions are obtained for the multiplicative functions in terms of hyperelliptic theta functions with zero arguments. In § 4 certain groups which are related to the group (0, 3; 2, 6, 6) are considered, and functions which are automorphic in these groups are expressed in terms of the functions already obtained.

§ 1. *The Riemann surface and the group associated with it.*

The Riemann surface considered, of genus $p = 3$, is defined by the equation

$$y^6 = (x-a)^2(x-b)^2(x-c)^3(x-d)^5. \tag{1}$$

The period paths are chosen as indicated in figures 1, 2, 3.

The three integrals

$$u_1 = \int \frac{dx}{y}, \qquad u_2 = \int \frac{(x-a)(x-b)(x-c)^2(x-d)^4 dx}{y^5},$$

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† *Automorphe Functionen*, Bd. 2, p. 136.

‡ See *Automorphe Functionen*, Bd. 1, p. 353, for this notation.

$$u_3 = \int \frac{(x-c)(x-d)\,dx}{y^2},$$

are of the first kind and linearly independent. Let A_i, B_i be the value of u_i

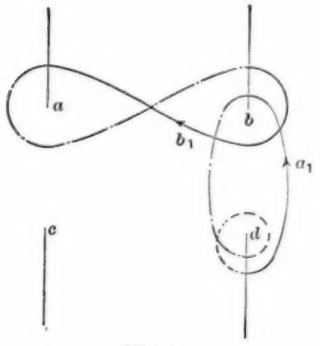

FIG. 1

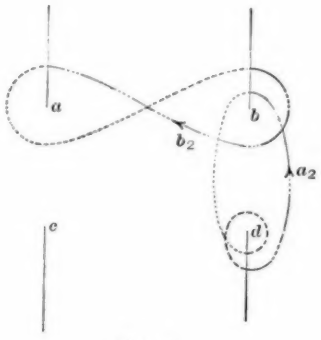

FIG. 2

when integrated around a_1 and b_1 respectively, and write ϵ for $e^{\pi i/3}$. Then the table of moduli of periodicity for these integrals is as follows:

	a_1	a_2	a_3	b_1	b_2	b_3
u_1	$-B_1$	B_1	$A_1-\epsilon^2 B_1$	A_1	$-A_1$	$\epsilon A_1 - B_1$
u_2	$-B_2$	B_2	$A_2+\epsilon B_2$	A_2	$-A_2$	$-\epsilon^2 A_2 - B_2$
u_3	$-B_3$	$-B_3$	$-A_3-\epsilon B_3$	A_3	A_3	$\epsilon^2 A_3 - B_3$

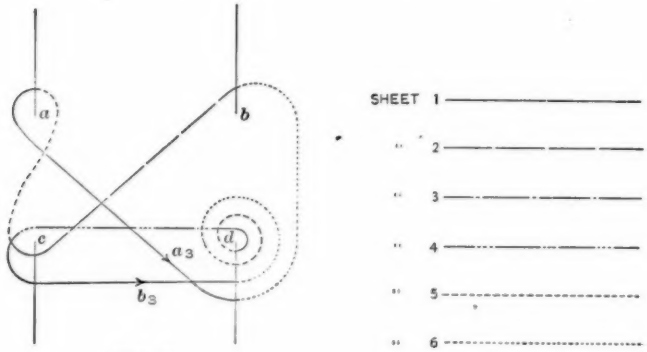

FIG. 3

Writing ω_i for the ratio $A_i : B_i$, the bilinear relations among the periods reduce to

$$\omega_2 = \frac{-\epsilon\omega_1 + 1}{\omega_1 - \epsilon^2}, \qquad \omega_3 = -\epsilon.$$

Instead of the system of moduli given in the preceding table, we introduce a simpler one by means of the canonical transformation

$$(2) \qquad \left|\begin{array}{ccc|ccc} 0 & 0 & 1 & & & \\ 0 & 1 & 0 & & 0 & \\ 1 & 1 & 0 & & & \\ \hline & & & 0 & 0 & 1 \\ & 0 & & -1 & 1 & 0 \\ & & & 1 & 0 & 0 \end{array}\right|.$$

Then, introducing the normal integrals v_1, v_2, v_3 in the usual way, the corresponding table of moduli for these will be:

$$(3) \qquad \left|\begin{array}{cccccc} v_1 & \pi i & 0 & 0 & \pi ia & -2\pi ib & \pi ib \\ v_2 & 0 & \pi i & 0 & -2\pi ia & 2\pi ia & -\pi ia \\ v_3 & 0 & 0 & \pi i & \pi ib & -\pi ia & \pi ic \end{array}\right|,$$

in which

$$a = \frac{\epsilon^2\zeta^2 + \epsilon}{1 - \zeta^2}, \qquad b = \frac{1 + \frac{\epsilon + \epsilon^2}{\sqrt{2}}\zeta - \zeta^2}{1 - \zeta^2}, \qquad c = \frac{\epsilon - \frac{1}{2}\zeta^2}{1 - \zeta^2};$$

$$(4) \qquad \zeta = \frac{\sqrt{2}}{\omega_1 - \epsilon^2}.$$

We now proceed to determine the effect on the moduli of a monodromy of the branch points. First, let a and b be interchanged by a deformation of the Riemann surface which shall rotate positively the branch line joining these two points. The period paths a_k, b_k will be deformed into new paths a_k', b_k'. Denote the value of u_i integrated along the old paths by $A_i^{(k)}$, $B_i^{(k)}$, respectively, and along the new paths by $A_i'^{(k)}$, $B_i'^{(k)}$. To express the new moduli in terms of the old, consider, for example, $A_i'^{(k)}$. Let T' denote the Riemann surface when cut along the paths a_k, b_k. Choose a closed path of integration for u_i in T' by starting from any point in a_k' and passing along this curve until it meets the boundary of T', then along the boundary to the point of a_k' on the opposite boundary, then along a_k' until it meets another boundary, and so on until the starting point is reached. The result of this integration is zero, whence, by transposing terms we obtain an expression for $A_i'^{(k)}$. The result is

$$A_i'^{(1)} = -B_i^{(1)} - B_i^{(3)}, \qquad B_i'^{(1)} = A_i^{(1)} + B_i^{(1)} + B_i^{(3)}.$$

From this follows

$$\omega_1' = \frac{A_1'^{(1)}}{B_1'^{(1)}} = -\epsilon^2\omega_1 - 1. \tag{5}$$

Next, let a describe a closed path in the positive direction about c. This gives the transformation

$$A_i'^{(1)} = A_i^{(1)}, \qquad B_i'^{(1)} = A_i^{(1)} - A_i^{(2)} + B_i^{(1)} + B_i^{(3)},$$

whence

$$\omega_1' = \frac{\omega_1}{\omega_1 + \epsilon}. \tag{6}$$

Corresponding to (5) and (6) we have for ζ the two transformations

$$\zeta' = \epsilon\zeta, \tag{7}$$

$$\zeta' = \frac{(\epsilon+1)\zeta - \epsilon^2\sqrt{2}}{\zeta\sqrt{2} - (\epsilon^2+1)}. \tag{8}$$

By operating first with (7), then with (8) we get

$$\zeta' = \frac{(\epsilon+\epsilon^2)\zeta - \epsilon^2\sqrt{2}}{\epsilon\zeta\sqrt{2} - (\epsilon+\epsilon^2)}. \tag{9}$$

Putting $\zeta = \eta/\sqrt{2}$ and $\zeta' = \eta'/\sqrt{2}$, (7) and (9) become

$$\eta' = \epsilon\eta; \qquad \eta' = \frac{(\epsilon+\epsilon^2)\eta - 2\epsilon^2}{\epsilon\eta - (\epsilon+\epsilon^2)}. \tag{10}$$

The transformations of the η-group are evidently of the form

$$\begin{vmatrix} A+iB & 2(C+iD) \\ (C-iD) & A-iB \end{vmatrix}$$

when reduced to determinate unity. The coefficients are of two types.

Type 1:

$$A = \frac{a}{2}, \qquad B = \frac{b\sqrt{3}}{2}, \qquad C = \frac{c}{2}, \qquad D = \frac{d\sqrt{3}}{2},$$

in which a, b, c, d are integers.

Type 2:

$$A = \frac{a\sqrt{3}}{2}, \qquad B = \frac{b}{2}, \qquad C = \frac{c\sqrt{3}}{2}, \qquad D = \frac{d}{2},$$

in which a, b, c, d are integers.

The coefficients in (10) are evidently of these types. That any combination of these substitutions will have coefficients of the same types follows from the fact that the coefficients of the generating substitutions (and hence of any substitution produced by them) are of the form $a + b\epsilon$, a and b being integers, and that every such substitution may be reduced to one of determinant unity by multiplying all the coefficients by some power of $i\epsilon$, giving a substitution of the first or second type according as the power of $i\epsilon$ is even or odd.

Where η is replaced by $\zeta\sqrt{2}$ it is seen that the ζ-transformations are of the two types

$$\begin{vmatrix} \frac{a+ib\sqrt{3}}{2} & \frac{c+id\sqrt{3}}{\sqrt{2}} \\ \frac{c-id\sqrt{3}}{\sqrt{2}} & \frac{a-ib\sqrt{3}}{2} \end{vmatrix}, \qquad \begin{vmatrix} \frac{a\sqrt{3}+ib}{2} & \frac{\sqrt{3}c+id}{\sqrt{2}} \\ \frac{c\sqrt{3}-id}{\sqrt{2}} & \frac{a\sqrt{3}-ib}{2} \end{vmatrix}.$$

Introduce a new variable z by means of the equation

$$\zeta = i\epsilon^2 \frac{z-i}{z+i}. \tag{11}$$

After dividing all the coefficients by $-2i$ and substituting $a = \alpha$, $b = \gamma$, $c + d = -2\beta$, $3c - d = 2\delta$, the transformation of the first type becomes

$$\text{(I)} \qquad \begin{vmatrix} \frac{\alpha+\beta\sqrt{6}}{2} & \frac{\gamma\sqrt{3}+\delta\sqrt{2}}{2} \\ \frac{-\gamma\sqrt{3}+\delta\sqrt{2}}{2} & \frac{\alpha-\beta\sqrt{6}}{2} \end{vmatrix},$$

in which

$$\alpha^2 + 3\gamma^2 - 2(3\beta^2 + \delta^2) = 4. \tag{12}$$

Making the same substitution in type 2, dividing all the coefficients by $-2i$, and substituting $a = \alpha$, $b = \gamma$, $d + 3c = -2\beta$, $d - c = 2\delta$ we obtain

$$\text{(II)} \qquad \begin{vmatrix} \frac{\alpha\sqrt{3}+\beta\sqrt{2}}{2} & \frac{\gamma+\delta\sqrt{6}}{2} \\ \frac{-\gamma+\delta\sqrt{6}}{2} & \frac{\alpha\sqrt{3}-\beta\sqrt{2}}{2} \end{vmatrix}$$

in which

$$3\alpha^2 + \gamma^2 - 2(\beta^2 + 3\delta^2) = 4. \tag{13}$$

Making the substitution (11) in (7) and (9) we have for the corresponding z-transformations

$$(14) \qquad z' = \frac{\frac{\sqrt{3}}{2}z + \frac{1}{2}}{-\frac{z}{2} + \frac{\sqrt{3}}{2}}, \qquad z' = \frac{\sqrt{2} - \sqrt{3}}{(\sqrt{2} + \sqrt{3})z}.$$

Denoting these by S and T respectively, we have now to show that *the group G generated by S and T consists of all the linear transformations contained in* (I) *and* (II).

It is evident that the fundamental region R of G may be taken to be the triangle bounded by the three circles (see Fig. 4)

$$(x \pm \sqrt{3})^2 + y^2 = 4,$$

$$x^2 + y^2 = (\sqrt{3} + \sqrt{2})^2 \qquad (x + iy = z).$$

It is further evident that G is contained within the group G' comprising all the substitutions (I) and (II), and hence the fundamental region R' of G' may be so chosen as to lie within R. It follows also from a general principle that R' may be so selected as to have only vertices that are fixed points for substitutions of the group. The mode of procedure would then be to write down the general expressions for the xy-coördinates of a fixed point, and subject them to the inequality conditions

$$(x \pm \sqrt{3})^2 + y^2 - 4 \leqq 0,$$

$$x^2 + y^2 - (\sqrt{3} + \sqrt{2})^2 \leqq 0,$$

these being the conditions that the point (x, y) shall be within or on the boundary of the triangle R. A somewhat lengthy analysis, but one that offers no particular difficulty, shows that the only fixed points of G' satisfying these conditions are the vertices of R itself [including the point $z = i(\sqrt{3} + \sqrt{2})$], and that all the substitutions of G' having these points as fixed points can be expressed as combinations of S and T.

Another form for the η-groups may be obtained by writing $a = 2\alpha + \beta$, $b = \beta$, $c = \gamma + \delta$, $d = \delta$ for substitutions of the first type, and $a = \beta$, $b = -2\alpha - \beta$, $c = \delta$, $d = -2\gamma - \delta$ for those of the second type whence it is seen that *the η-group consists of all the transformations of the two types*

$$\begin{vmatrix} \alpha + \beta\epsilon & 2(\gamma + \delta\epsilon) \\ \gamma - \delta\epsilon^2 & \alpha - \beta\epsilon^2 \end{vmatrix}, \qquad \begin{vmatrix} \alpha + \beta\epsilon & 2(\gamma + \delta\epsilon) \\ -\gamma + \delta\epsilon^2 & -\alpha + \beta\epsilon^2 \end{vmatrix}$$

in which $\alpha, \beta, \gamma, \delta$ are integers subject to the condition

$$\alpha^2 + \alpha\beta + \beta^2 - 2(\gamma^2 + \gamma\delta + \delta^2) = 1.$$

§ 2. *The multiplicative functions.*

A function $\phi(\zeta)$ having the property

$$\phi\left(\frac{\alpha_k\zeta+\beta_k}{\gamma_k\zeta+\delta_k}\right)=\mu_k(\gamma_k\zeta+\delta_k)^d\phi(\zeta) \tag{15}$$

for all the substitutions of a given group, will be called a *multiplicative* function of degree d, with multipliers μ_k. We shall consider only functions without poles in the fundamental region.

To consider for the moment a more general case than the one under consideration let the fundamental region be a circle-arc polygon entirely within the fixed circle, whose vertices A_i are fixed points for elliptic substitutions of period l_i. Every function $\phi(\zeta)$ of given degree d will have a zero at A_i of an order not less than a certain number λ_i. There will also be a certain number N of simple zeros within (or in special cases on the boundary of) the region. This number is readily found to be*

$$N=\tfrac{1}{2}d(n-1)-\tfrac{1}{2}\sum_i\frac{d+2\lambda_i}{l_i},$$

in which n is the number of pairs of conjugate sides, and the summation relates to the different cycles of vertices.

To determine the value of λ_i, take one of the vertices $\zeta=a$ of the ith cycle. The elliptic substitution having a for fixed point may be written in the form

$$\frac{\zeta'-a}{\zeta'-a'}=e^{\frac{2\pi i}{l_i}}\frac{\zeta-a}{\zeta-a'},$$

in which a' is the other fixed point. Let μ_i be the multiplier corresponding to this substitution. Following the method of POINCARÉ† let a new function $\psi(\zeta)$ be defined by the relation

$$\psi(\zeta)=(\zeta-a')^d\phi(\zeta).$$

Then

$$\psi(\zeta')=\mu_i e^{-d\pi i/l_i}\psi(\zeta). \tag{16}$$

Assuming

$$\psi(\zeta)=\sum_{\nu=\lambda_i}^{\infty}A_\nu\left(\frac{\zeta-a}{\zeta-a'}\right)^\nu,$$

and substituting in (16) we find on equating the coefficients of the lowest powers in both members,

$$\mu_i=e^{\pi i(2\lambda_i+d)/l_i}. \tag{17}$$

* Cf. FORSYTH, *Theory of functions*, § 307.

† Acta Mathematica, vol. 1 (1882), p. 193.

The degree d can be even, but cannot be odd for every group.

To show this, take the special case in which the fundamental region is a quadrilateral formed by two symmetrically situated triangles having like angles $\pi/l_i (i = 1, 2, 3)$, and let the triangles have in common the side adjacent to the angles π/l_1, π/l_3. Denote by S_1, S_2, S_3 the substitutions of periods l_1, l_2, l_3 having the vertices of one of the triangles of the quadrilateral for fixed points. Then $S_3 S_1 S_2 = 1$ and hence $\mu_1 \mu_2 \mu_3 = 1$, from which follows,

$$\sum_{i=1}^{3} \frac{2\lambda_i + d}{l_i} \equiv 0 \pmod 2.$$

Assume $l_i = 2^{\rho_i} l_i'$ in which l_i' is odd and ρ_i is a positive integer or zero. Then the preceding relation becomes

$$\sum_i 2^{\rho_{i+1}+\rho_{i+2}} l_{i+1}' l_{i+2}' (2\lambda_i + d) = 2^{\rho_1+\rho_2+\rho_3} \cdot 2c,$$

in which c is an integer, and the subscripts are to be reduced (mod. 3) when they exceed 3. It is evident from this formula that d can be odd only when two of the exponents $\rho_{i+1} + \rho_{i+2}$ are equal and are less than the third. Suppose $\rho_i + \rho_j = \rho_i + \rho_k < \rho_j + \rho_k$. From this follows $\rho_j = \rho_k > \rho_i$. Hence, *the degree d can be odd only when two of the exponents $\rho_i (i = 1, 2, 3)$ are equal, and are greater than the third exponent.*

In the case of the ζ-group, $l_1 = l_2 = 6$, $l_3 = 2$, and accordingly $\rho_1 = \rho_2 = \rho_3 = 1$. Hence d cannot be odd. Let $d = 2m$. Then from (17)

$$\mu_i = e^{\pi i(\lambda_i + m)/3} \qquad (i = 1, 2),$$

$$\mu_3 = e^{\pi i(\lambda_3 + m)}.$$

The simplest cases are:

(*a*) $m = 1$, $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$; the function $\phi(\zeta)$ has a simple zero at e_1 (Fig. 4)* and at congruent points, and vanishes nowhere else.

(*b*) $m = 1$, $\lambda_2 = 1$, $\lambda_1 = \lambda_3 = 0$; $\phi(\zeta)$ has a simple zero at e_2 and at congruent points.

(*c*) $m = 3$, $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 0$; $\phi(\zeta)$ has a simple zero at e_3 and at congruent points.

The three different functions corresponding to these cases will be denoted by ϕ_1, ϕ_2, ϕ_3 respectively. It will be the object of the next section to prove that these functions actually exist.

* This statement is used for brevity to mean that $\phi(\zeta)$ has a zero at the point in the ζ-plane which corresponds to the point e_1 by means of the transformation (11).

§ 3. *Transformation of the theta functions.*

The table of periods for u_1, u_2, u_3 (§ 1) can be reduced in an infinity of ways to the form

$$\left|\begin{array}{ccc|ccc} \omega_{11} & \omega_{12} & 0 & \omega'_{11} & \omega'_{12} & 0 \\ \omega_{21} & \omega_{22} & 0 & \omega'_{21} & \omega'_{22} & 0 \\ 0 & 0 & \omega_{33} & 0 & 0 & \omega'_{33} \end{array}\right|,$$

and hence the theta functions having the moduli of table (3) can be reduced to hyperelliptic theta functions in an infinity of different ways. A convenient reduction of this kind is obtained thus. Let the theta function with arguments zero be represented by the formula

$$\vartheta\begin{bmatrix} g \\ h \end{bmatrix}(\zeta) = \sum_{m_1, m_2, m_3} e^{\sum_{\kappa,\lambda}^{1,2,3} a_{\kappa\lambda}(m_\kappa + g_\kappa)(m_\lambda + g_\lambda) + \pi i \sum_{\mu=1}^{3} h_\mu (m_\mu + g_\mu)}, \tag{18}$$

in which

$$a_{11} = 2a_{22} = -a_{23} = \pi i a, \quad -2a_{12} = a_{13} = \pi i b, \quad a_{33} = \pi i c.$$

Introducing new letters of summation n_1, n_2, n_3 by means of the substitution

$$n_1 = m_1, \qquad n_2 = 2m_2 - m_3, \qquad n_3 = m_3,$$

the right member of formula (18) becomes

$$\vartheta\begin{bmatrix} g_1 & g_2 - g_3/2 \\ h_1 & h_2 \end{bmatrix}(\zeta)\,\vartheta\begin{bmatrix} g_3/2 \\ h_2 + 2h_3 \end{bmatrix}(2\epsilon) + \vartheta\begin{bmatrix} g_1 & (1 + 2g_2 - g_3)/2 \\ h_1 & h_2 \end{bmatrix}(\zeta)\,\vartheta\begin{bmatrix} (1+g_3)/2 \\ h_2 + 2h_3 \end{bmatrix}(2\epsilon),$$

in which the hyperelliptic theta functions have the moduli a_{11}, a_{12}, a_{22} and the elliptic theta functions have the modulus 2ϵ, the arguments being zero in all cases. We proceed now to determine the effect of the transformations S and T, formulæ (7) and (9) on the functions $\vartheta\left[\begin{smallmatrix} g_1 & g_2 \\ h_1 & h_2 \end{smallmatrix}\right](\zeta)$.

To the transformation S on ζ corresponds the transformation

$$\left|\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & -2 & 0 & 0 \\ 0 & 4 & 2 & 1 \end{array}\right|$$

on the theta moduli. This leads to the relation

$$(19)\quad \frac{1}{\sqrt{1-\zeta'^2}}\vartheta\begin{bmatrix} -\frac{1}{2}+g_1-2g_2+h_1 & g_2 \\ -g_1+2g_2 & -4g_2+2h_1+h_2 \end{bmatrix}(\zeta') = \frac{1}{\sqrt{1-\zeta^2}}\, e^{-\pi i[g_1^2-4g_1g_2+8g_2^2+2g_1h_1-4g_2h_1-\frac{1}{12}]}\vartheta\begin{bmatrix} g_1 & g_2 \\ h_1 & h_2 \end{bmatrix}(\zeta),$$

in which $\zeta' = \epsilon\zeta$. The signs of the radicals are chosen so as to be alike when $\zeta = 0$.

To T corresponds the transformation

$$\left|\begin{array}{cc|cc} 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ \hline -2 & 0 & -1 & -1 \\ 0 & -4 & -2 & -1 \end{array}\right|$$

on the theta moduli and hence the relation

$$(20)\quad \frac{1}{\sqrt{1-\zeta'^2}}\vartheta\begin{bmatrix} g_1+2g_2-2h_1-h_2 & g_1+g_2-h_1-h_2 \\ 2g_1-h_1-h_2 & 4g_2-2h_1-h_2 \end{bmatrix}(\zeta') = \frac{\sqrt{2}\epsilon\zeta-(\epsilon+\epsilon^2)}{\sqrt{1-\zeta^2}}\, e^{\pi iE}\vartheta\begin{bmatrix} g_1 & g_2 \\ h_1 & h_2 \end{bmatrix}(\zeta),$$

in which

$$\zeta' = \frac{(\epsilon+\epsilon^2)\zeta-\epsilon^2\sqrt{2}}{\sqrt{2}\epsilon\zeta-(\epsilon+\epsilon^2)}$$

and

$$E = 2g_1^2+8g_1g_2+4g_2^2-8g_1h_1-4g_1h_2-8g_2h_1-8g_2h_2+4h_1^2+6h_1h_2+2h_2^2+\tfrac{1}{2}.$$

The signs of the radicals are to be alike when $\zeta' = \zeta$.

For brevity we write

$$\frac{1}{\sqrt{1-\zeta^2}}\vartheta\begin{bmatrix} g_1g_2 \\ h_1h_2 \end{bmatrix}(\zeta) = \Phi\begin{bmatrix} g_1g_2 \\ h_1h_2 \end{bmatrix}.$$

Then by means of formulæ (19) and (20) it is easy to verify that the functions ϕ_1, ϕ_2, ϕ_3 referred to at the end of the preceding section may be expressed in the form

$$\phi_2 = \Phi\begin{bmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & -\frac{1}{3} \end{bmatrix}\Phi\begin{bmatrix} -\frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 0 \end{bmatrix},$$

$$\phi_1 = \frac{1}{\phi_2}\,\Phi\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\Phi\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\Phi\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\Phi\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

$$\phi_3 = \Phi\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\Phi\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}\Phi\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}\Phi\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}\Phi\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\Phi\begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}.$$

It is necessary to show that these functions do not vanish identically. For this purpose consider the value of the function $\Phi\left[\begin{smallmatrix} -\frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 0 \end{smallmatrix}\right]$ when ζ is zero. It readily reduces to

$$e^{\frac{7\pi i}{72}} \sum_{n_1=-\infty}^{+\infty} e^{-\frac{\pi\sqrt{3}}{2}(n_1-\frac{1}{6})^2+\frac{\pi i}{2}(n_1^2-n_1)} \cdot \sum_{n_2=-\infty}^{+\infty} e^{-\pi\sqrt{3}(n_2+\frac{1}{6})^2}.$$

The last factor is evidently not zero. By writing for brevity $e^{-\pi\sqrt{3}/72} = t$, the first series may be put in the form

$$t(1-t^{48}-t^{288}) + \sum_{n=1}^{\infty}\left[t^{(24n-1)^2} - t^{(24n+17)^2}\right]$$

$$+ \sum_{n=1}^{\infty}\left[t^{(24n+1)^2} - t^{(24n+7)^2}\right] + t^{25}(1-t^{96}-t^{144})$$

$$+ \sum_{n=1}^{\infty}\left[t^{(24n+5)^2} - t^{(24n+11)^2}\right] + \sum_{n=1}^{\infty}\left[t^{(24n-5)^2} - t^{(24n+13)^2}\right].$$

Since $t^{48} = 0.02658\cdots$, it follows that all the terms in this expression are positive, and hence the function does not vanish. Since the two factors Φ in ϕ_2 are interchanged by S, it follows that if one does not vanish identically, the other does not. Similarly it may be proved that ϕ_1 and ϕ_3 do not vanish identically.

Every multiplicative function (without poles) can be expressed as a product of the functions ϕ_1, ϕ_2, ϕ_3 and functions of the form $a\phi_1^6 + b\phi_2^6$, a, b being constants. Every automorphic function belonging to the group can be expressed in the form

$$\prod_{i=1}^{n} \frac{a_i\phi_1^6 + b_i\phi_2^6}{c_i\phi_1^6 + d_i\phi_2^6}.$$

§ 4. *The z-group and its relation to the ternary quadratic form.*

By comparison with the results of FRICKE-KLEIN (*Automorphe Functionen*, pp. 533–538) it is evident that the substitutions of the z-group (§ 1) correspond to linear transformations of the ternary form $2z_1^2 - z_1z_2 - 3z_3^2$ into itself. Not all of the corresponding ternary substitutions have integer coefficients, but the z-group is *commensurable** with the group whose corresponding ternary substitutions have integer coefficients. Denote the latter group by Γ'. According to the general theory† the substitutions of Γ' are comprised in eight different types, which, on account of the fact that p is even reduces to only four distinct types, viz.,

* Two groups are said to be *commensurable* when they have a common invariant subgroup of finite index under each.

† We use in what follows the notation and results of FRICKE-KLEIN, loc. cit.

I

$$\begin{vmatrix} \frac{a+b\sqrt{6}}{2} & \frac{c\sqrt{3}+d\sqrt{2}}{2} \\ \frac{-c\sqrt{3}+d\sqrt{2}}{2} & \frac{a-b\sqrt{6}}{2} \end{vmatrix},$$

II

$$\begin{vmatrix} \frac{a\sqrt{3}+b\sqrt{2}}{2} & \frac{c+d\sqrt{6}}{2} \\ \frac{-c+d\sqrt{6}}{2} & \frac{a\sqrt{3}-b\sqrt{2}}{2} \end{vmatrix},$$

III

$$\begin{vmatrix} \frac{a\sqrt{6}+b}{2} & \frac{c\sqrt{2}+d\sqrt{3}}{2} \\ \frac{-c\sqrt{2}+d\sqrt{3}}{2} & \frac{a\sqrt{6}-b}{2} \end{vmatrix},$$

IV

$$\begin{vmatrix} \frac{a\sqrt{2}+b\sqrt{3}}{2} & \frac{c\sqrt{6}+d}{2} \\ \frac{-c\sqrt{6}+d}{2} & \frac{a\sqrt{2}-b\sqrt{3}}{2} \end{vmatrix},$$

in which a, b, c, d are integers subject to the conditions,

(1) that each determinant is unity, and

(2) that a, c are even for substitutions of types I and II, and b, d are even for the other two types.

That the number of types is diminished by half when p is even is readily shown as follows. Suppose $p = 2\pi_1 p_2$, in which π_1, p_2 are relatively prime. Then, corresponding to the division of p into the two factors $2\pi_1$, p_2, we have the type

$$\begin{vmatrix} \frac{a\sqrt{2\pi_1 r_1}+b\sqrt{p_2 r_2}}{2}\sqrt{q_1} & \frac{c\sqrt{2\pi_1 r_2}+d\sqrt{p_2 r_1}}{2}\sqrt{q_2} \\ \frac{-c\sqrt{2\pi_1 r_2}+d\sqrt{p_2 r_1}}{2}\sqrt{q_2} & \frac{a\sqrt{2\pi_1 r_1}-b\sqrt{p_2 r_2}}{2}\sqrt{q_1} \end{vmatrix},$$

the coefficients of which must satisfy the conditions

$$2\pi_1 r_1 q_1 a^2 - p_2 r_2 q_1 b^2 + 2\pi_1 r_2 q c^2 - p_2 r_1 q_2 d^2 = 4,$$

$$a 2\pi_1 q_1 r_1 \equiv b p_2 q_1 r_2 \equiv c 2\pi_1 q_2 r_2 \equiv d p_2 q_2 r_1 \pmod 2.$$

As p, q, r are relatively prime, p_2, q_1, r_2, q_2, r_1 are not divisible by 2 and hence b and d are even. Write $b = 2b'$, $d = 2d'$ and the above determinant becomes

$$\begin{vmatrix} \frac{a\sqrt{\pi_1 r_1}+b'\sqrt{2p_2 r_2}}{\sqrt{2}}\sqrt{q_1} & \frac{c\sqrt{\pi_1 r_2}+d'\sqrt{2p_2 r_1}}{\sqrt{2}}\sqrt{q_2} \\ \frac{-c\sqrt{\pi_1 r_2}+d'\sqrt{2p_2 r_1}}{\sqrt{2}}\sqrt{q_2} & \frac{a\sqrt{\pi_1 r_1}-b'\sqrt{2p_2 r_2}}{\sqrt{2}}q_1 \end{vmatrix}.$$

But these substitutions are the same as those of the type in which $s = 2$, and p

is separated into two factors π_1 and $2\rho_2$, while q, r are factored in the same way as before. Also, the determinant of the one type reduces to that of the other.

The same line of argument may be followed when r is even. Hence, *if p or r is even the number of distinct types is* $\leqq T_p T_q T_r$.

Suppose now that condition (2) is removed, and let the enlarged group be denoted by Γ. The z-group is then a subgroup of finite index under Γ consisting of all the substitutions of types I and II. We will denote this group by (I, II). Similarly we define the subgroups (I, III), (I, IV).

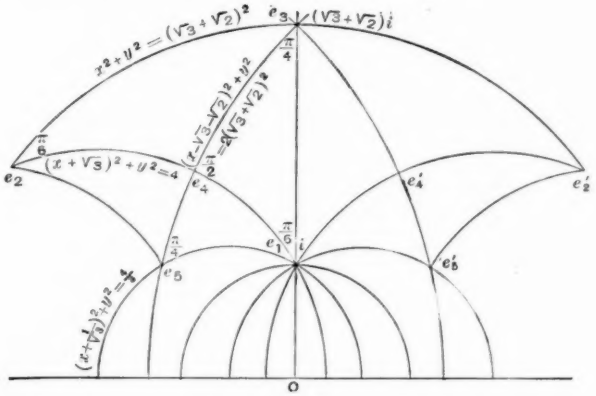

FIG. 4.

The fundamental regions for these groups are shown in Fig. 4, in which $e_1e_4e_3e_4'$ is the fundamental region for Γ with e_1e_4 conjugate to e_1e_4' and e_3e_4' conjugate to e_3e_4; $e_1e_2e_2'$ is the fundamental region for (I, II) with e_1e_2 and e_3e_2 conjugate to e_1e_2' and e_3e_2' respectively; $e_1e_5e_3e_5'$ is the fundamental region for (I, III) with e_1e_5, $e_4'e_5'$, e_4e_5 conjugate respectively to e_1e_5', $e_4'e_3$, e_4e_3; and finally, $e_1e_5e_3e_5'$ is the fundamental region for (I, IV) with e_1e_5 and e_3e_5' conjugate to e_1e_5' and e_3e_5.

In all these groups the substitutions of type I form an invariant subgroup of finite index which will be denoted by I. The fundamental region for I is $e_1e_5e_2e_2'e_5'$, in which e_3e_2, e_1e_5, e_5e_2 are conjugate to e_3e_2, e_1e_5, $e_2'e_2'$. The group I when represented by ζ-substitutions evidently consists of all operations formed by combinations of S and T in which S occurs an even number of times. Hence the group I may be generated by S^2, T, and STS^{-1}. Referring to section 2 it is easy to verify that, since I is of genus zero, every function which is automorphic for the group is expressible rationally in terms of $(\phi_1/\phi_2)^3$. Functions belonging to Γ are rational in $\tau=\phi_3^4/\phi_1^6\phi_2^6$, and those belonging to I, III are rational in $\sqrt{\tau}=\phi_3^2/\phi_1^3\phi_2^3$. Finally, functions belonging to (I, IV) are rational in $\sqrt{\tau+c}$, in which c is a non-vanishing con-

stant. To prove the last statement, let $\lambda_1 = \phi_1^3 \phi_2^3$ and λ_2 be two particular multiplicative functions for the group (I, IV) such that every multiplicative function having a single zero within the fundamental region is expressible in the form $a_1\lambda_1 + a_2\lambda_2$. Suppose λ_2 is that particular function which is 0^4 at e_3 (Fig. 4). Then ϕ_3^4 being a multiplicative function for Γ is expressible in the form $\lambda_2(a_1\lambda_1 + a_2\lambda_2)$, in which a_1, a_2 are chosen so that the second factor is 0^4 at e_5 and at points congruent to it in the group (I, IV). Solving the equation $\lambda_2(a_1\lambda_1 + a_2\lambda_2) = \phi_3^4$ for the ratio $\lambda_1 : \lambda_2$ we find that it is expressible in the form $a + b\sqrt{\tau + c}$, and hence, every function belonging to (I, IV) is rational in terms of $\sqrt{\tau + c}$, since it is rational in $\lambda_2 : \lambda_1$. Conversely, every function which is rational in τ, $\sqrt{\tau}$, or $\sqrt{\tau + c}$ is automorphic in Γ, (I, III), or (I, IV) respectively.

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A THEOREM CONCERNING THE INVARIANTS OF LINEAR HOMOGENEOUS GROUPS, WITH SOME APPLICATIONS TO SUBSTITUTION-GROUPS*

BY

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1. Consider a group G of order N, written in linear homogeneous form in terms of the variables $x_1, x_2, \cdots, x_n$. Let the multipliers of S_i, any substitution of G, be $\theta_1, \theta_2, \cdots, \theta_n$, and designate by χ_i the "characteristic"† $\theta_1 + \theta_2 + \cdots + \theta_n$ of S_i. We shall now prove the

LEMMA. *If*

$$\sum_{i=1}^{N} \chi_i \neq 0,$$

then will G have an absolute invariant, linear in $x_1, x_2, \cdots, x_n$. ‡

Let $t_1, t_2, \cdots, t_n$ be arbitrary constants. The expression

$$(S_1 + S_2 + \cdots + S_N)(t_1 x_1 + t_2 x_2 + \cdots + t_n x_n) \equiv I$$

will be an absolute invariant of G of degree 1, if it does not vanish identically. If it does, then must the coefficients of the terms $t_1 x_1, t_2 x_2, \cdots, t_n x_n$ of I each be $= 0$. Accordingly, if the *sum* of these coefficients is $\neq 0$, then $I \not\equiv 0$. But this sum is readily found to be

$$\sum_{i=1}^{N} \chi_i.$$

Hence the lemma.

We shall now prove the following

THEOREM. *The number of linearly independent absolute invariants of G of the first degree in $x_1, x_2, \cdots, x_n$ is equal to*

$$\frac{1}{N} \sum_{i=1}^{N} \chi_i.$$

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† BURNSIDE, *On the Representation of a Group of Finite Order as an Irreducible Group of Linear Substitutions*, etc., Proceedings of the London Mathematical Society, November, 1903, pp. 117–123. The quantities χ_i have been called *weights* by the author in two papers published in these Transactions, vol. 4 (1903), p. 387, and vol. 5 (1904), p. 310.

‡ Cf. equation (true for transitive groups) given by BURNSIDE, Proceedings of the London Mathematical Society, March, 1903, p. 122, bottom.

According to the preceding lemma, G possesses an invariant $I = z_1$ if $\sum \chi_i \neq 0$. Then, by a theorem concerning the reducibility of linear groups,* G is intransitive, its systems being z_1 and (say) $z_2, z_3, \cdots, z_n$. Let the group in the variables $z_2, z_3, \cdots, z_n$ be denoted by G', and its characteristics by χ_i'. The multipliers of z_1 all being $= 1$, we have $\sum \chi_i = N + \sum \chi_i'$.

Conversely, if G possesses an invariant $I = z_1$, then $\sum \chi_i \neq 0$. For, writing G as an intransitive group in terms of $z_1, z_2, z_3, \cdots, z_n$, we have $\sum \chi_i = N + \sum \chi_i'$. Now, if $\sum \chi_i' \neq 0$, we have an absolute invariant of degree 1 in $z_2, z_3, \cdots, z_n$, and G' must be intransitive, its variables breaking up into two systems containing, respectively, 1 and $n-2$ variables. We would find that $\sum \chi_i' = N + \sum \chi_i''$. Continuing thus, we see the truth of the theorem stated.

COROLLARY 1. *The number of linearly independent absolute invariants of G of degree m in $x_1, x_2, \cdots, x_n$ is equal to*

$$\frac{1}{N} \sum_{i=1}^{N} \chi_i^{(m)},$$

where $\chi_i^{(m)}$ represents the sum of the homogeneous products of $\theta_1, \theta_2, \cdots, \theta_n$ of degree m.

We prove the corollary by constructing from G the group whose variables are all the different terms of degree m in the letters $x_1, x_2, \cdots, x_n$, and then applying the theorem given. In case the group constructed (which is isomorphic with G) is of order N/k, we write each of its substitutions k times before applying the theorem, regarding the entire set of N substitutions as forming a group.

2. Let it be proposed to find an upper limit to the least order of existing absolute invariants of a given group G. We shall take as an illustration a group in 3 variables. The quantity $\chi^{(m)}$ is here represented by

$$(1) \qquad \begin{vmatrix} 1 & 1 & 1 \\ \theta_1 & \theta_2 & \theta_3 \\ \theta_1^{2+m} & \theta_2^{2+m} & \theta_3^{2+m} \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 \\ \theta_1 & \theta_2 & \theta_3 \\ \theta_1^2 & \theta_2^2 & \theta_3^2 \end{vmatrix} \equiv A_0\theta_1^m + B_0\theta_2^m + C_0\theta_3^m,$$

if θ_1, θ_2 and θ_3 are all different; in the case $\theta_1 = \theta_2 \neq \theta_3$ we get (by treating (1) as an indeterminate form)

$$\chi^{(m)} = A_1\theta_1^m + B_1 m\theta_1^m + C_1\theta_3^m,$$

and, if $\theta_1 = \theta_2 = \theta_3$, we obtain

$$\chi^{(m)} = A_2\theta_1^m + B_2 m\theta_1^m + C_2 m(m-1)\theta_1^m;$$

the quantities A_i, B_i and C_i being independent of m.

*MASCHKE, Mathematische Annalen, vol. 52 (1899), p. 363; LOEWY, these Transactions, vol. 4 (1903), p. 44.

Substituting in $N^{-1}\sum\chi_i^{(m)} = k$ we obtain an equation, the right-hand member of which is an integer representing the number of invariants of degree m of G. Collecting, in the left-hand member, all the terms having the same factors θ_i^m, $m\theta_i^m$ or $m(m-1)\theta_i^m$, this member takes the form

$$\sum_i \theta_i^m K_i + m\sum_i \theta_i^m L_i + m(m-1)\sum_i \theta_i^m M_i, \tag{2}$$

where the coefficients K_i, L_i and M_i are independent of m.

A root θ, occurring never more than once among the multipliers of the different substitutions of the group, will appear in only one term $(\theta^m K)$ in (2); a root θ, occurring twice as a multiplier in some one of the substitutions of G, will appear as a term $\theta^m K$ and as a term $m\theta^m L$ in the expression (2); etc. Accordingly, there are as many terms in (2) as there are different multipliers in G, each multiplier counted once, twice or three times, according as it occurs once, twice or three times in some one or other of the substitutions of G (e. g., the root 1 must be counted three times, occurring three times in the identical substitution). Let us designate this number by I'.

Taking now in turn $m = 1, 2, 3, \cdots, I'$, we have I' equations, the right-hand members of which cannot all be zero unless the determinant obtained by eliminating the quantities K_i, L_i and M_i is zero (the quantities K_i, L_i and M_i are not all zero, as their sum $= N$, as shown by putting $m = 0$). Now, this determinant is readily found to be the product of the first, second, or third powers of the differences of the various roots θ and does, therefore, not vanish. Accordingly, there must be at least one absolute invariant of G whose order is not greater than I'. Again, it is evident that $I' \leqq 3$ times the number of conjugate sets of substitutions of G. Hence, generalizing our results we see that a *linear homogeneous group G in n variables must have at least one absolute invariant whose order is not greater than n times the number of conjugate sets of substitutions of G.*

3. Having given two isomorphic groups, we may construct an intransitive group and employ the corollary to show the existence or not of invariant bilinear functions of the two sets of variables. Thus, let the two isomorphic groups be G and G in the variables $x_1, x_2, \cdots, x_n$ and $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$, G being obtained from G by replacing in the latter every variable and coefficient by its conjugate imaginary expression. Then will

$$\frac{1}{N}\{\Sigma\chi_i^{(2)} + \Sigma\chi_i\bar{\chi}_i + \Sigma\bar{\chi}_i^{(2)}\}$$

be the total number of absolute invariants of degree 2 of the intransitive group exhibited by

$$\begin{array}{c|c} G & 0 \\ \hline 0 & \bar{G} \end{array},$$

and

$$\frac{1}{N}\Sigma\chi_i\bar{\chi}_i \tag{3}$$

will evidently be the number of invariants of the form

$$\sum_{i,j} a_{i,j} x_i \bar{x}_j .$$

There must be at least one such, as not all the terms of (3) are zero (e. g., that from S_1 (identity) $= n^2$), and none of them are negative. It is well known that there is always one such invariant, a positive-definite Hermitian form,* and BURNSIDE has shown that if G is transitive, there is only one invariant of the form

$$\sum_{i,j} a_{i,j} x_i \bar{x}_j .\dagger$$

Hence, *for a transitive group* G, $N^{-1}\Sigma\chi_i\bar{\chi}_i = 1$, *whereas, for an intransitive group,* $N^{-1}\Sigma\chi_i\bar{\chi}_i > 1$.‡

4. COROLLARY 2. *Let* ϕ_i *be any integral symmetric function with integral coefficients of the multipliers of the substitution* S_i *of the group* G *of order* N. *Then is*

$$\sum_{i=1}^{N} \phi_i = kN,$$

where k *is an integer.*

We prove this corollary first for the case when ϕ_i is one of the elementary symmetric functions of the multipliers $\theta_1, \theta_2, \cdots, \theta_n$. Suppose

$$\phi_i = \theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3 + \cdots .$$

Let G' be obtained from G merely by replacing the variables $x_1, x_2, \cdots, x_n$ by $y_1, y_2, \cdots, y_n$. Then the group H, whose variables are $x_iy_j - x_jy_i$ $(i, j = 1, 2, \cdots, n;\ i \neq j)$ will be isomorphic with G, and its characteristics are the quantities ϕ_i. If its order is N/k, we write each of its substitutions k times. Then

$$\sum_{i=1}^{N} \phi_i = k_1 N$$

by the theorem stated above. §

Next, by considering an intransitive group constructed from two such groups,

* FUCHS, Berliner Sitzungsberichte (1896), p. 753; MOORE, Chicago University Record (1896), Mathematische Annalen, vol. 50 (1898), p. 213; LOEWY, Comptes Rendus (1896), p. 168.

† Proceedings of the London Mathematical Society, November, 1903, p. 121.

‡ Ibid., p. 123.

§ Cf. BURNSIDE, Proceedings of the London Mathematical Society, Nov., 1900, p. 162.

H_1 and H_2, whose characteristics are elementary symmetric functions ϕ_i' and ϕ_i'' (the substitutions of each being repeated such a number of times that the total number equals the order of the isomorphic group G), we prove that $\sum \phi_i' \phi_i'' = k_2 N$. This process may be continued, and we prove finally that

$$\sum_{i=1}^{N} \sigma_i = kN,$$

where σ_i is any function of the elementary symmetric functions of the multipliers $\theta_1, \theta_2, \cdots, \theta_n$.

Now, any integral symmetric function with integral coefficients of a certain number of quantities $\theta_1, \theta_2, \cdots, \theta_n$ can be expressed as an integral function with integral coefficients of the elementary symmetric functions of the same quantities. The corollary follows.

5. A substitution-group G in n letters $x_1, x_2, \cdots, x_n$ may be written as a linear homogeneous group G' whose variables are the n letters given. The multipliers belonging to a substitution S of G can be written down immediately by the following rule: to each of the letters $x_1, x_2, \cdots, x_n$ left unchanged by S belongs a multiplier 1, to a cycle of S of order a belongs the a different ath roots of unity. The characteristic of S, i. e., the sum of the multipliers is therefore equal to the number of letters left unchanged by S. Accordingly, by the theorem of § 1, *the average number of letters which remain unchanged by a substitution of a group G is equal the number of linear invariants of G;* i. e., is equal the number of transitive sets of G.* This theorem may be readily extended by Corollary 2 by taking for ϕ_i the sum of the 2d, 3d, $\cdots$, powers of the multipliers of the corresponding substitution of G. We find immediately that *the total number of letters occurring in all of the cycles of a given order m is a multiple of the order of the group.*†

6. A number of theorems of like nature may evidently be proved by purely arithmetical means, and will be more or less different according to the mode in which the given group is represented. We shall here mention only two such, one by FROBENIUS and one by MAILLET.

If n is a factor of N, the number of substitutions of G satisfying the equation $S^n = 1$ is a multiple of n.‡

Write G as a regular group of degree N. If h_i represents the total number of substitutions of G of order i, then must, according to Corollary 2, every coefficient of t in

$$(1+t)^N + h_2(1-t^2)^{N/2} + h_3(1+t^3)^{N/3} + h_4(1-t^4)^{N/4} + \cdots = P$$

* FROBENIUS, Journal für Mathematik, vol. 101 (1887), p. 288.

† NETTO, Journal für Mathematik, vol. 103 (1888), p. 321.

‡ FROBENIUS, Berliner Sitzungsberichte, 1895, pp. 984–985.

be a multiple of N. The coefficient of t^m in $(1 \pm t^k)^{N/k}$ is, namely, the elementary symmetric function of degree m of the multipliers of a substitution of order k of the group as written.

Noticing that, if $N = p^a m = kpl$, we have

$$(1 \pm t^k)^{N/k} \equiv (1 \pm t^{kp})^{N/kp} \pmod{p^a} \qquad (p \text{ a prime})$$

and then also

$$\begin{aligned} P \equiv\ & (1 \pm t^{p^a})^{N/p^a}(1 + h_p + h_{p^2} + \cdots + h_{p^a}) \\ & + (1 \pm t^{n_1 p^a})^{N/n_1 p^a}(h_{n_1} + h_{n_1 p} + \cdots + h_{n_1 p^a}) \\ & + (1 \pm t^{n_2 p^a})^{N/n_2 p^a}(h_{n_2} + h_{n_2 p} + \cdots + h_{n_2 p^a}) \\ & \cdots\cdots\cdots\cdots\cdots \\ & + (1 \pm t^{n_0 p^a})^{N/n_0 p^a}(h_{n_0} + h_{n_0 p} + \cdots + h_{n_0 p^a}) \\ & + h_b(1 \pm t^b)^{N/b} + h_c(1 \pm t^c)^{N/c} + \cdots \pmod{p^a}, \end{aligned}$$

where $n = p^a n_0$ (p prime to n_0) and is not divisible by $b, c, \cdots$; and where $1, n_1, n_2, \cdots, n_0$ are all the different factors of n_0, in ascending order of magnitude. By considering in turn the coefficients of $t^{p^a}, t^{n_1 p^a}, \cdots, t^{n_0 p^a}$ we find that they are all $\equiv 0 \pmod{p^a}$. It follows that the number of substitutions satisfying $S^n = 1$ is $\equiv 0 \pmod{p^a}$ and is therefore also $\equiv 0 \pmod{n}$.

*If G contains substitutions which leave unchanged $n_0 = n$ letters (identity), n_1 letters, $\cdots$, n_r letters, this list containing all the substitutions, then is the order N of G a factor of $(n_0 - n_1)(n_0 - n_2)\cdots(n_0 - n_r)$.**

Let the number of substitutions of each kind be respectively $h_0 = 1, h_1, h_2, \cdots, h_r$. Then, by choosing for ϕ_i of Corollary 2 in turn $1, \chi_i, (\chi_i)^2, (\chi_i)^3, \cdots$ we obtain the congruences

$$\begin{aligned} h_0 + h_1 + h_2 + \cdots + h_r &= N, \\ n_0 h_0 + n_1 h_1 + n_2 h_2 + \cdots + n_r h_r &\equiv 0, \\ n_0^2 h_0 + n_1^2 h_1 + n_2^2 h_2 + \cdots + n_r^2 h_r &\equiv 0 \pmod{N}. \\ \cdots\cdots\cdots\cdots\cdots & \end{aligned}$$

Hence,

$$h_i \prod_{\substack{j=0,1,\ldots,r; \\ j \neq i}} (n_i - n_j) \equiv 0 \pmod{N},$$

which includes the theorem stated.

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* MAILLET, Annales de la Faculté des Sciences de Toulouse, 1895, p. 8.

ON THE GEOMETRY WHOSE ELEMENT IS THE 3-POINT OF A PLANE*

BY

F. MORLEY

The study of the 3-points of a plane is a part of the study of the cubic curves of the plane, but it is a part of special interest.

The sections 1–3 of this investigation, dealing with fully and triply perspective triangles, are mainly recapitulatory. The mapping of § 3 is discussed synthetically by KANTOR.† But it is so much more easily grasped by means of equations that I have not scrupled to repeat a part of KANTOR's argument, with dualistic apparatus. Passing to unrestricted 3-points, the mapping is not carried out, for it seems necessary first to work out (§§ 4, 5) a cubic curve arising from two 3-points, and this curve leads (§§6, 7) to a phenomenon which seems fundamental, namely, that the 3-points of a plane fall in general into sets of three.

§ 1. *Fully perspective triangles.*

Three points of a plane are called a 3-point. The case of points on a line is degenerate and is excluded unless specially mentioned.

We denote a 3-point by a Roman capital, and there will be no ambiguity in denoting the triangle of the 3-point, and the 3-line of the 3-point, by the same capital, though a stricter notation would be that of a Greek capital for the 3-line.

We note first the fully perspective triangles, that is triangles sixway perspective. These are well known as arising from the flex-configuration of a cubic curve. They occur in *tetrads*, or in sets of four.

The polar points of any line as to the 4 are on a line and form a self-apolar set.

§ 2. *Triply perspective triangles.*

It is well known that if $a_1a_2a_3$ is perspective with $b_1b_2b_3$ and with $b_2b_3b_1$ then it is perspective with $b_3b_1b_2$. A convenient proof is to identify the theorem with the theorem that a 3-point has a circumcenter.

* Read in part at the Boston Summer Meeting, 1903. Received for publication May 7, 1904.

† Crelle's Journal, vol. 95 (1883), p. 147.

For if the antipoints of x_2x_3 are a_1b_1, of x_3x_1 are a_2b_2, of x_1x_2 are a_3b_3, $x_1x_2x_3$ being real points, then a_1b_3, a_2b_1, a_3b_2 meet at a circular point I, and a_1b_2, a_2b_3, a_3b_1 meet at J; but on the other hand a_1b_1, a_2b_2, a_3b_3 meet at the circumcenter of $x_1x_2x_3$. It is worth remarking that this theorem is identical with Pascal's theorem for a two-line.*

It is convenient to think of triply perspective triangles as concentric equilateral triangles. That the centers of perspection form a triangle mutual with the two is then obvious; that is, triply perspective 3-points fall into sets of three, or *triads*.

It is further obvious from this metrically canonic case that two triangles A, B fully perspective with a given one T are triply perspective, for if T contain I and J then A, B are equilateral and concentric.

Fig. 1, which is KANTOR's configuration 9_3A, shows a triad of 3-points, and

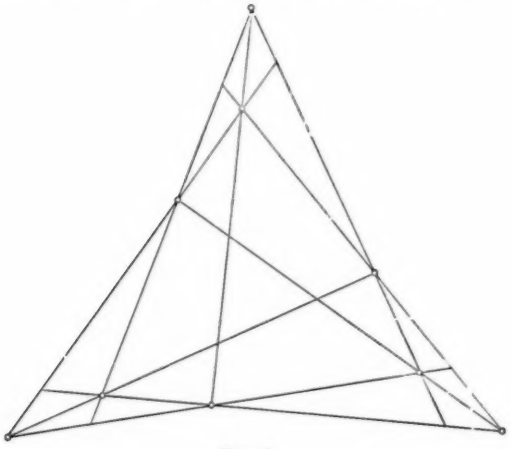

FIG. 1.

also the accompanying triad of 3-lines. Note that there is no such thing as a triad of *triangles* in the present sense.

The coördinates of a 3-point fully perspective with a 3-point T of reference are, if $\omega = \exp.(2\pi i/3)$,

2. 1) $$A: \begin{bmatrix} a_1 & a_2 & a_3 \\ a_1 & \omega a_2 & \omega^2 a_3 \\ a_1 & \omega^2 a_2 & \omega a_3 \end{bmatrix}.$$

Three such 3-points ABC form a triad when

2. 2) $$\begin{aligned} a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 &= 0, \\ a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1 &= 0; \end{aligned}$$

* Cf. SCHROETER, Mathematische Annalen, vol. 2 (1870), p. 533.

We note from adding these equations that collinear points of ABC are apolar with T.

A triad of triply perspective 3-points obviously belongs to a range of cubic line-curves, whence observing that the equation of A is

2. 3) $$a_1^3\xi_1^3 + a_2^3\xi_2^3 + a_3^3\xi_3^3 - 3a_1a_2a_3\xi_1\xi_2\xi_3 = 0,$$

we infer that the equations

2. 4) $$\begin{vmatrix} a_1^3 & a_2^3 & a_3^3 & a_1a_2a_3 \\ b_1^3 & b_2^3 & b_3^3 & b_1b_2b_3 \\ c_1^3 & c_2^3 & c_3^3 & c_1c_2c_3 \end{vmatrix} = 0$$

are deducible from (2. 2).

Let A and B coincide. Then the 3-point C is inscribed in the 3-line A. Incidentally it is worth noting how, given A and the inscribed 3-point C, the 3-point T fully perspective with A and C is determined. Let A be the reference-triangle, T be

$$\begin{bmatrix} t_1 & t_2 & t_3 \\ t_1 & \omega t_2 & \omega^2 t_3 \\ t_1 & \omega^2 t_2 & \omega t_3 \end{bmatrix}$$

and C be

$$\begin{bmatrix} 0 & \lambda & 1 \\ 1 & 0 & \mu \\ \nu & 1 & 0 \end{bmatrix}.$$

Then they are perspective in the order given if

$$\lambda t_1 t_3^2 + \omega\mu t_2 t_1^2 + \omega^2 \nu t_3 t_2^2 = \mu\nu t_1 t_2^2 + \omega^2 \nu\lambda t_2 t_3^2 + \omega\lambda\mu t_3 t_1^2,$$

and they are perspective in all ways if

$$\lambda t_1 t_3^2 = \mu t_2 t_1^2 = \nu t_3 t_2^2,$$

or if

$$t_1 = \sqrt[3]{\nu^2\lambda}, \qquad t_2 = \sqrt[3]{\lambda^2\mu}, \qquad t_3 = \sqrt[3]{\mu^2\nu}.$$

Hence *two triangles, one inscribed to the other, may always be projected into equilateral triangles.* It is true that any two triangles may be projected into equilateral triangles, but the investigation does not belong here.

§ 3. *Transference to a cubic surface.*

We consider now the equation

3. 1) $$\Xi_1 x_1^3 + \Xi_2 x_2^3 + \Xi_3 x_3^3 + \Xi_0 x_1 x_2 x_3 = 0,$$

connecting a point x of a plane S_2 and a plane Ξ of a space S_3. Given x, Ξ is on a point X, with coördinates

3. 2) $$X_i = x_i^3, \qquad X_0 = x_1 x_2 x_3 \qquad (i = 1, 2, 3),$$

so that the S_2 maps into the cubic surface Φ:

3. 3) $$X_1 X_2 X_3 = X_0^3.$$

This surface is in the BRILL-SCHILLING collection (ser. 7, no. 9).

The 3 lines $X_0 = X_i = 0$ whose points are on Φ will be called rays of Φ; the other 3 lines of reference, whose planes are on Φ, will be called axes of Φ.

But (3. 2), if true for x_1, x_2, x_3, is equally true for $x_1, \omega x_2, \omega^2 x_3$. Hence a point of Φ represents a 3-point of S_2, fully perspective with the 3-point of reference T.

Given Ξ, x is on a cubic curve ϕ with flexes on the lines of T.

Thus a plane section of Φ maps into a cubic curve, the correspondence being 1:3. The collinear points ABC of Φ map into the 9 meets of a pencil of cubics ϕ. Such a pencil of cubics contains (besides the 3-line T) three 3-lines. Hence, collinear points ABC map into triply perspective 3-points. When A and B coincide, C is on S_2 a 3-point on the lines of A. Hence *a tangent plane of* Φ *represents a* 3*-line of* S_2. This may be verified directly; for the 3-line A is

$$\frac{x_1^3}{a_1^3} + \frac{x_2^3}{a_2^3} + \frac{x_3^3}{a_3^3} - \frac{3x_1x_2x_3}{a_1a_2a_3} = 0,$$

whence

$$\Xi_i = \frac{1}{a_i^3}, \qquad \Xi_0 = -\frac{3}{a_1a_2a_3},$$

and

3. 4) $$27\,\Xi_1\Xi_2\Xi_3 + \Xi_0^3 = 0,$$

the plane-equation of Φ.

The tangent plane of Φ at A being

$$X_1a_1^3 + X_2a_2^3 + X_3a_3^3 = 3X_0a_1a_2a_3,$$

we see that to the tangent planes on a point X correspond 3-lines of a line-cubic

3. 5) $$X_1\xi_1^3 + X_2\xi_2^3 + X_3\xi_3^3 = 3X_0\xi_1\xi_2'\xi_3$$

and beginning with this we could reverse the argument of this section.

To the 3 points and 3 planes of Φ on any line corresponds Figure 1.

Since a 3-point and its 3-line are represented by a point of Φ and its tangent plane, to the *triangle* of the plane corresponds the *element* (point and plane thereat) of Φ. The triangle of reference is represented by the point on the axes, and the plane on the rays. The point on the axes will be called T.

To 2 triangles in 4-fold perspective correspond 2 elements whose points are on a plane with an axis of the surface, or (what follows) whose planes are on a point with the opposite ray of the surface.

In particular if a point X of Φ represents the points of one of a system of equilateral triangles with common center, then the middle points of the sides may be represented by the point

$$-\tfrac{1}{2}X_0,\ -\tfrac{1}{8}X_1,\ X_2\ \ X_3.$$

§ 4. *The Clebschian of triply perspective 3-points.*

The tetrads of fully perspective triangles (§ 1) such as $TA_1A_2A_3$ are represented by points $TA_1A_2A_3$ on a line, A_i being on Φ. Since the polar of T is a repeated plane, these 4 points are self-apolar.

Thus a plane on T meets the cubic in a cubic curve with the same property; any tangent from T is therefore a stationary tangent, for when of 4 self-apolar points two coincide, three do. Such a cubic curve is sometimes called equianharmonic; as its form is reducible to the sum of 3 cubes, it may be called *catalectic*, after SYLVESTER's use of catalecticant. It is of course one for which the invariant S of degree 4 vanishes; from the present position this is because (1), the polar conic of T is a repeated line cutting out flexes F_i; (2), hence the polar conic of F_i is two lines on T; (3), hence the 4 tangents of the cubic at and from F_i are self-apolar.

If I may call the cubic $(abx)^3$ derived from $(a\xi)^3$ and $(b\xi)^3$ their *Clebschian* —a name of course covering other like cases—then the Clebschian of the 3-points A and B is

$$\begin{vmatrix} a_1^3 & a_2^3 & a_3^3 \\ b_1^3 & b_2^3 & b_3^3 \\ x_1^3 & x_2^3 & x_3^3 \end{vmatrix} = 0.$$

This is a catalectic cubic apolar to A and B; *it is represented simply by the plane* TAB.

It is equally the Clebschian of the cubics of the web built on T, A, B; to express its equation symmetrically in terms of the triad ABC we have from (2. 4)

$$\begin{aligned} a_2^3 b_3^3 - a_3^3 b_2^3 &= \rho(a_1^3 \cdot b_1 b_2 b_3 - b_1^3 \cdot a_1 a_2 a_3) \\ &= \rho a_1 b_1 (a_1^2 b_2 b_3 - b_1^2 a_2 a_3) \end{aligned}$$

or from (2. 2)

$$= \rho' a_1 b_1 c_1.$$

Hence the Clebschian is

$$a_1 b_1 c_1 x_1^3 + a_2 b_2 c_2 x_2^3 + a_3 b_3 c_3 x_3^3 = 0.$$

It is worth noting that if on a catalectic cubic ϕ we take collinear points ABC and join to one of the Cayleyan points, we get 6 other points on ϕ which lie on two lines. For if we cut

$$x_1^3 + x_2^3 + x_3^3 = 0$$

by $(\beta x) = 0$ and join to $(0\,0\,1)$ we get

$$\beta_3^3(x_1^3 + x_2^3) = (\beta_1 x_1 + \beta_2 x_2)^3,$$

and the points where these lines meet ϕ lie on the 3 lines

$$(\beta_3 x_3)^3 + (\beta_1 x_1 + \beta_2 x_2)^3 = 0,$$

one of course being β itself. These 3 lines meet on the Hessian line $x_3 = 0$. Thus if

$$TA_1A_2A_3, \qquad TB_1B_2B_3, \qquad TC_1C_2C_3,$$

are tetrads of fully perspective 3-points, of which $A_1B_1C_1$ are a triad then (when the ordering is right) $A_2B_2C_2$ and $A_3B_3C_3$ are also triads and the Clebschian of any two of the 3-points $A_iB_jC_k$ is the Clebschian of every two.

§ 5. *The Clebschian of any two* 3*-points.*

Hitherto only 3-points fully perspective with a given one have been considered. With regard to two unrestricted 3-points I wish to prove here one proposition; I hope to discuss the general question in another paper. The proposition arises from the Clebschian.

The Clebschian of any two cubics,

5. 1, 1a) $\qquad X_1\xi_1^3 + X_2\xi_2^3 + X_3\xi_3^3 + 6X_0\xi_1\xi_2\xi_3 = 0, (a\xi)^3 = 0,$

contains no term in $x_1x_2x_3$. Hence calling the reference 3-point *syzygetic* with (5. 1), the four 3-points syzygetic with either cubic are apolar to the Clebschian; as of course are also the cubics themselves. In particular let $X_1 = X_2 = X_3 = 0$; then the Clebschian of

$$\xi_1\xi_2\xi_3 = 0, \qquad (a\xi)^3 = 0,$$

is

5. 2) $\qquad (a_2x_3 - a_3x_2)(a_3x_1 - a_1x_3)(a_1x_2 - a_2x_1) = 0,$

and the 3-point $\xi_1\xi_2\xi_3$ is both on and apolar to the Clebschian. Thus the Clebschian of two 3-points is subject to the 8 conditions of having two such 3-points given, and to a further condition.

If the 8 conditions are independent, there is a pencil of such cubics, and they meet in a third 3-point. This is to be investigated in §§ 6 and 7. The 8 con-

ditions are not independent if the two 3-points are triply perspective, for then, as is easily verified, a cubic on both and apolar to one is apolar to the other.

And if the two 3-points are fully perspective, any cubic on both is apolar to both; and *the Clebschian is the whole plane.*

A cubic ϕ being given by

$$x = p(u), \qquad y = p'(u),$$

the condition that a 3-point u_i be both on and apolar is seen by reference to HALPHEN (*Fonctions Elliptiques*, vol. 2, p. 428) to be

5. 3) $$\zeta(u_2 - u_3) + \zeta(u_3 - u_1) + \zeta(u_1 - u_2) = 0,$$

whence *the projection of such a 3-point from a point of ϕ on to ϕ is again such a 3-point*, and the sides of such a 3-point cut out another one. By independent proof of the former result the last formula is seen at once. For let u_1 be the flex 0, then u_2 and u_3 are apolar with the polar conic of the flex, i. e., $p'(u_2) = p'(-u_3)$ and this is another form of (5. 3) when $u_1 = 0$.

On the Clebschian of two given 3-points u_i and v_i a large number of points can be at once constructed.

Thus if we complete the 3-point $v_1 ab$ fully perspective with u_i, and cross-join ab and $v_2 v_3$ we obtain two points of the curve—12 points in all.

And if we take a point e, such that ev_2, ev_3 are the Hessian of the lines eu_i* then e is a point of ϕ.

And if the conic $u_1 u_2 u_3 v_2 v_3$ meets ϕ again at c, and $u_1 u_2 u_3 v_2 v_3 d$ are apolar along the conic, then cdv_1 are on a line.

And the points where $u_2 u_3$ and $u_i v_j$ meet ϕ may be readily determined.

§6. *Three-lines on and apolar to a deltoid.*

Taking now the pencil of cubics of §5, we select a rational curve of the pencil. For convenience of statement† I take this rational cubic as a deltoid, Δ^3,

$$t^3 - xt^2 + yt - 1 = 0.$$

The question is first: When is a 3-line on this deltoid apolar with it? The line-equation of the curve, given by $\xi_1 = t^3 - 1$, $\xi_2 = -t^2$, $\xi_3 = t$, is

$$\xi_2^3 + \xi_3^3 = \xi_1 \xi_2 \xi_3.$$

Hence lines $\xi\eta\zeta$ are apolar with it when

$$6(\xi_2 \eta_2 \zeta_2 + \xi_3 \eta_3 \zeta_3) = |\sum \xi_1 \eta_2 \zeta_3,$$

*See Bulletin of the American Mathematical Society, vol. 1 (1895), p. 124.

† For a sketch of the vector treatment I refer to my paper on *Orthocentric properties of the plane n-line*, Transactions, vol. 4 (1903), pp. 1–12.

and if the parameters of these lines are given by

$$t_i^3 - s_1 t_i^2 + s_2 t_i - s_3 = 0,$$

then

$$6(s_3^2 - s_3) = \Sigma t_1^2 t_2 (t_3^3 - 1) = (s_3 - 1)\Sigma t_1^2 t_2;$$

that is, discarding the case of lines on a point for which $s_3 = 1$,

$$6s_3 = \Sigma t_1^2 t_2,$$

or

$$s_1 s_2 = 9s_3. \qquad 6.\ 1)$$

Now s_1 is the circumcenter; hence *the circumcenter of the 3-line is on the cusp-circle.*

There are then in the deltoidal case no real non-degenerate 3-lines both on

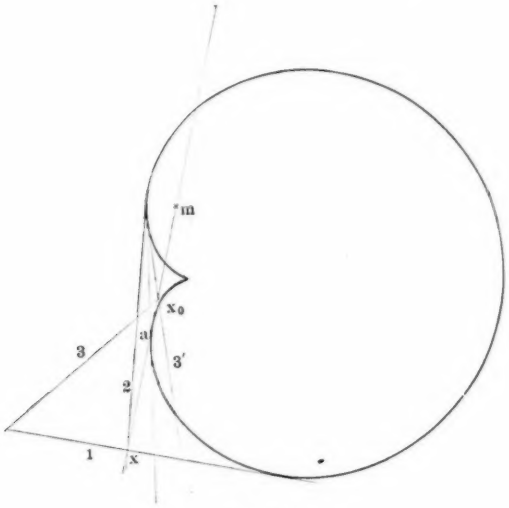

FIG. 2.

and apolar. There are such in the cardioidal case; to construct them we write the equation for t_3:

$$t^2(t_1 + t_2) + t_1 t_2 (t_1 + t_2) + t(t_1^2 + t_2^2 - 6t_1 t_2) = 0,$$

where

$$tt' = t_1 t_2 = 1/t_0,$$

say. Then

$$(t + t' + t_1 + t_2)(t_1 + t_2) = s/t_0,$$

so that if x is the known point $t_0 + t_1 + t_2$ where 2 of the 3-lines meet, x_0 is the point $t_0 + t + t'$ on both of the solution-lines, and $x + x_0 = 2m$, then

6. 2) $$(m - t_0)(x - t_0) = 4/t_0,$$

an involution with double points where the line t_0 meets the curve again.

Fig. 2 shows two such 3-lines, namely, 123 and 123′.

§ 7. *The triplicity of 3-points.*

Let a_i and b_i be two such 3-lines of the deltoid. Their Clebschian is

$$\sum_{(b)} |\xi a_1 b_1||\xi a_2 b_2||\xi a_3 b_3| = 0$$

summed for cyclical permutations of b_i. The common lines of this and Δ^3 are given by

$$\sum_{(b)} |t^3 - 1,\ a_1^2,\ b_1||t^3 - 1,\ a_2^2,\ b_2||t^3 - 1,\ a_3^2,\ b_3| = 0.$$

But

$$\begin{vmatrix} t^3 - 1 & t^2 & t \\ a^3 - 1 & a^2 & a \\ b^3 - 1 & b^2 & b \end{vmatrix} = (tab - 1)(t - a)(t - b)(a - b).$$

Hence the third 3-line is given by

7. 1) $$\sum_{(b)} (ta_1 b_1 - 1)(ta_2 b_2 - 1)(ta_3 b_3 - 1)(a_1 - b_1)(a_2 - b_2)(a_3 - b_3) = 0,$$

or if

$$I = \Sigma(a_1 - b_1)(a_2 - b_2)(a_3 - b_3) = 3(q_3 - r_3) - q_2 r_1 + q_1 r_2,$$

where q_i, r_i are the coefficients of a_i and b_i, e. g., $q_1 = a_1 + a_2 + a_3$, then

$$(t^3 q_3 r_3 - 1) I - t^2\{\ \} + t\Sigma(a_1 b_1 + a_2 b_2 + a_3 b_3)(a_1 - b_1)(a_2 - b_2)(a_3 - b_3) = 0.$$

Hence

$$q_3 r_3 s_3 = 1,$$

$$I s_2/s_3 = (q_3 - r_3) q_1 r_1 - 3 q_3 r_1^2 + 3 r_3 q_1^2 + 9(q_3 r_2 - r_3 q_2) - q_2 r_2 (q_1 - r_1),$$

or, since $9q_3 = q_1 q_2$, $9r_3 = r_1 r_2$,

$$I s_2/s_3 = q_1 r_1 [q_3 - r_3 - \tfrac{1}{3}(q_2 r_1 - q_1 r_2)] = \tfrac{1}{3} I q_1 r_1.$$

Thus either $I = 0$, that is the 3-lines are apolar to each other along the curve, in which case the algebra shows that the Clebschian is Δ^3, or *in general*

$$s_2/s_3 = q_1 r_1/3,$$

whence

7. 2) $$q_1 r_1 s_1 = 27 .$$

That is, the circumcenters, which we know are on the cusp-circle, are along that circle apolar with the cusps.

Hence the relation of the three 3-lines is mutual, and returning to the dualistic statement:

Two general 3-points of a plane determine uniquely and mutually a third; or the 3-points of a plane projectively considered fall into sets of three — say form a *triplicity*. An exceptional case is that of two fully perspective 3-points — these are *neutral*, i. e., determine no 3-point. Another apparently exceptional case is that of two triply perspective 3-points, but it is likely that by knowing further properties of the triplicity we should find that the triads of § 2 do belong to it.

JOHNS HOPKINS UNIVERSITY,
April, 1904.

SUFFICIENT CONDITION FOR A MINIMUM WITH RESPECT TO ONE-SIDED VARIATIONS*

BY

GILBERT AMES BLISS

In a problem of the calculus of variations with fixed endpoints, the curves along which the integral is taken may be free to traverse any portion of the plane, or they may be restricted to lie in a given region R. In the latter case it sometimes happens that although no solution can be found which lies entirely within the region, yet one exists consisting partly of interior arcs and partly of arcs of the boundary. In the neighborhood of the boundary arcs the only variations considered must be of course interior to the region, and therefore lie on one side only of the curve whose minimizing properties are to be investigated.

The necessary conditions which such a combination of interior curve and boundary must satisfy in order to minimize the given integral have already been derived. It is proposed in the following paper to prove that these conditions with slight modification are also sufficient: first, when the integral has the form

$$I = \int f(x, y, y')\,dx; \tag{1}$$

and second, when the integral is

$$I = \int F(x, y, x', y')\,dt, \tag{2}$$

taken along a curve

$$x = \mu(t), \quad y = \nu(t)$$

in parameter representation.

PART I. THE PROBLEM FOR $I = \int f(x, y, y')\,dx$.

§ 1. *The necessary conditions.*

Consider an integral in the form (2) for which the function f is of class C''',† and $\partial' f/\partial y'^2 > 0$,‡ in a region including in its interior all (x, y, y') points which satisfy

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† A function is said to be of class $C^{(i)}$ when it is continuous and has continuous derivatives up to and including those of order i. It is of class $D^{(i)}$ when the derivatives mentioned have only a finite number of discontinuities (compare BOLZA, *Lectures on the Calculus of Variations*, The University of Chicago Press, Chicago, 1904, p. 7). Similarly a curve is of class $C^{(i)}$ or $D^{(i)}$ if the functions which represent it have the corresponding properties.

‡ Problems in which f has this property are called *regular*.

$$R': \quad \begin{cases} (x, y) \text{ in a region } R \text{ of the } xy \text{ plane,} \\ |y'| \leqq L, \end{cases}$$

where L is a positive constant. At least part of the boundary of R is composed of a curve of class D'.

Consider a curve E joining two given points a and b interior to R, and consisting for convenience of only three pieces E_1, B, E_2, each of class C''; the pieces E_1 and E_2 are interior to R except at the points c and d where they

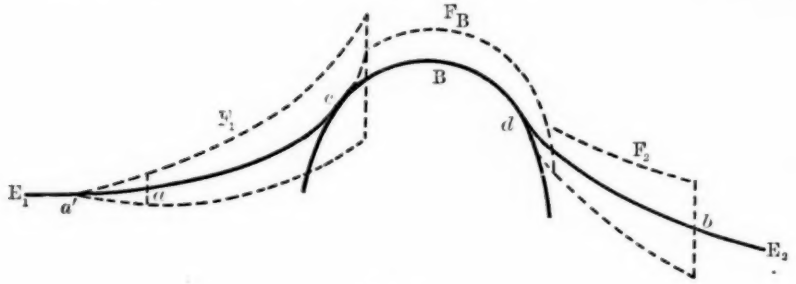

FIG. 1.

adjoin the arc B of the boundary of R, as in Fig. 1. If this curve is to minimize I the following necessary conditions must be satisfied:

I) E_1 and E_2 must be extremals, i. e. they must satisfy Euler's equation,

$$f_y - \frac{d}{dx} f_{y'} = 0\,; \tag{3}$$

II) No conjugate point to a lies between a and c on E_1, and no conjugate point to d lies between d and b on E_2;†

III) At c and d, E_1 and E_2 are respectively tangent to B;‡

IV) Along B,

$$f_y - \frac{d}{dx} f_{y'} \leqq 0.\S$$

*See for example, OSGOOD, *Sufficient Conditions in the Calculus of Variations*, Annals of Mathematics, 2d Ser., vol. 2 (1900), p. 108; also BOLZA, loc. cit., p. 22. The literal subscripts x, y, y' denote partial differentiation.

† BOLZA, loc. cit., pp. 63, 67.

‡ BOLZA, loc. cit., p. 42 footnote. The usual form of this condition is

$$E(x, y, y', \beta') = f(x, y, \beta') - f(x, y, y') - (\beta' - y') f_{y'}(x, y, y') = 0 \text{ at } c \text{ and } d,$$

where β' and y' refer to the curves B and the extremals E_1, E_2 respectively. But for a regular problem it follows from Taylor's formula that E can vanish only when $y' = \beta'$.

§ BOLZA, loc. cit., p. 43.

It has been tacitly assumed that the region R lies above the curve B. Otherwise condition IV would read $\leqq 0$.

If the conditions II and IV are amended to read

II′) No conjugate point to a lies between a and c or at c on E_1, and no conjugate point to d lies between d and b or at b on E_2;

IV′) Along B,

$$f_y - \frac{d}{dx} f_{y'} > 0;$$

then I, II′, III, IV′ *are sufficient conditions for a minimum.* The proof of the last statement is the object of the present paper.

§ 2. *Existence theorems for Euler's equation.*

The differential equation (3), discovered by Euler, which characterizes the extremals for a problem of the calculus of variations of the form stated in § 1, is of the second order. Since $f_{y'y'} > 0$, it can be solved for y'' so that

$$y'' = e(x, y, y'), \tag{4}$$

where the function e is of class C' in the region including R' in which the properties of f were supposed to hold. In the sequel two existence theorems for this equation will be used, which are stated here without proof.*

The essence of the first theorem is that if one solution of equation (4) exists in the region R' through an initial point and direction (ξ_0, η_0, η_0'), it can always be surrounded by an infinity of other solutions all defined on the same interval and through neighboring initial elements (ξ, η, η'). In precise language, if

$$y = \Phi(x) \tag{5}$$

is a solution of equation (4) of class C'' on an interval $[\xi_0 \xi_1]$, with initial values (ξ_0, η_0, η_0'), and entirely interior to R', then a function

$$y = \phi(x, \xi, \eta, \eta') \tag{6}$$

and a positive constant ρ exist with the following properties:

1) The function ϕ and its x-derivatives ϕ', ϕ'' are of class C' for all values of x, ξ, η, η' satisfying the inequalities,

$$|\xi - \xi_0| \leqq \rho, \ |\eta - \eta_0| \leqq \rho, \ |\eta' - \eta_0'| \leqq \rho, \ \xi_0 - \rho \leqq x \leqq \xi_1 + \rho;$$

2) The curve $y = \phi$ solves equation (4) and takes the initial element (ξ, η, η') at $x = \xi$:

* See PICARD, *Traité d'Analyse*, vol. 2, chap. 11; PEANO, Atti di Torino, vol. 33 (1897), p. 9; and the references in the *Encyclopädie der Mathematischen Wissenschaften*, IIA 4a, pp. 195, 200.

3) No other curve of class C'' on the interval $[\xi_0 - \rho, \xi_0 + \rho]$ has the properties 2).

On account of 3) it is seen that the curve (5) is represented by (6) when $(\xi, \eta, \eta') = (\xi_0, \eta_0, \eta_0')$.

Let P′ denote any perfect region of points (ξ, η, η') in R'. The second theorem states that one and only one solution of equation (4) exists through each element (ξ, η, η') in P′, and gives a minimum interval in which such a solution is surely defined: i. e., there always exists corresponding to P′ a function

$$y = \phi(x, \xi, \eta, \eta')$$

and a positive constant ρ with the properties:

1) The function ϕ and its x-derivatives ϕ', ϕ'' are of class C' in the region defined by the conditions,

$$(\xi, \eta, \eta') \text{ in } \mathrm{P}', \quad \xi \leqq x \leqq \xi + \rho;$$

2) The curve $y = \phi$ is a solution of equation (4) with the initial element (ξ, η, η') when $x = \xi$;

3) No other curve of class C'' on $[\xi, \xi + \rho]$ has the properties 2).

§ 3. *The field.*

Suppose that the four conditions I, II′, III, IV′ of § 1 are satisfied by the curve E. Then a set of extremals exist all passing through a point a' on the extremal E_1, which simply cover a portion of the plane, a so-called field, about the curve E_1. Let the element of E_1 at a' be $\xi, \eta, \bar{\eta}'$ with $\xi < x_a$.

If a' is near enough to a the former of the two existence theorems stated above, with ξ_0, η_0, η_0' replaced by x_a, y_a, y_a',* shows that there is a set of extremals

$$y = \phi(x, \xi, \eta, \eta') = \psi_1(x, \eta'), \tag{7}$$

through the point a', and such that $\psi_1, \psi_1', \psi_1''$ are of class C' for

$$|\eta' - \bar{\eta}'| \leqq \rho, \qquad x_a \leqq x \leqq x_e + \delta, \tag{8}$$

where ρ and δ are two suitably chosen positive quantities.

As is well known the x-coördinate of the point conjugate to (ξ, η) on one of the extremals (7), is defined by a zero of the derivative $\partial\phi/\partial\eta'$.† When $(\xi, \eta, \eta') = (x_a, y_a, y_a')$ this derivative determines the conjugate to a on the extremal E_1 and on account of IV′ can not vanish anywhere except at a on the arc ac.

* The literal subscripts a, b, c, d designate the coördinates of the corresponding points.

† BOLZA, loc. cit., p. 62.

It follows that, if a', δ, ρ are properly chosen, the derivative $\partial\psi_1/\partial\eta'$ can not vanish anywhere in the region (8). Namely from § 1, 2),

$$\eta = \phi(\xi, \xi, \eta, \eta'), \qquad \eta' = \phi'(\xi, \xi, \eta, \eta'),$$

and therefore

$$0 = \frac{\partial\phi(\xi, \xi, \eta, \eta')}{\partial\eta'}, \qquad 1 = \frac{\partial\phi'(\xi, \xi, \eta, \eta')}{\partial\eta'},$$

in particular when $(\zeta, \eta, \eta') = (x_a, y_a, y_a')$. Hence on account of the continuity properties of ϕ, an interval $[x_a - h, x_a + h]$ can be found such that the derivative $\partial\phi'/\partial\eta'$ is different from zero for all arguments (x, ξ, η, η') for which x is in the interval, a' on the corresponding arc of E_1, and η' the direction of E_1 at a'. That is, $\partial\phi(x, \xi, \eta, \eta')/\partial\eta'$ vanishes only at $x = \xi$ in the interval $[x_a - h, x_a + h]$. Let now a' approach a on E_1. Then it must be possible to take a' so near to a, and a constant δ so small that the latter derivative is different from zero for all values of x in $[x_a + h, x_c + \delta]$. Otherwise on account of continuity considerations $\partial\phi(x, x_a, y_a, y_a')/\partial y_a'$ would vanish and the point a would have a conjugate on the arc ac, which is contrary to hypothesis. But since $\partial\phi(x, \xi, \eta, \eta')/\partial\eta'$ is different from zero when η' is the direction of E_1 at a', and x satisfies the second inequality (8), it must also be different from zero for all points satisfying (8) when ρ is properly restricted.

The field is the region F_1 of the xy-plane bounded by the four curve pieces which correspond to the boundary of the region (8) (see Fig. 1). That these curves are non-intersecting and divide the plane into an interior and an exterior, follows because the derivative $\partial\psi_1/\partial\eta'$ is never zero, and consequently on any ordinate ψ_1 increases (decreases) from $\psi(x, \bar{\eta}' - \rho)$ to $\psi(x, \bar{\eta}' + \rho)$ as η' increases from $\bar{\eta}' - \rho$ to $\bar{\eta}' + \rho$. It is seen from this also that through each point of F_1, there passes one and but one extremal of the set (7). The function

$$\eta' = \eta'(x, y)$$

so defined in the region F_1 is of class C', since in the vicinity of any point of F_1 it is the solution of the equation

$$y = \psi_1(x, \eta'),$$

which has continuous partial derivatives of the first order.*

In a manner somewhat similar a field F_B can be constructed above the curve B by means of a set of extremals each tangent to B. Let the equation of this arc of the boundary be

$$y = \beta(x).$$

* For the theorems on implicit functions see JORDAN, *Cours d'Analyse*, vol. 1, p. 80.

By the second existence theorem, with the totality of points and directions of B as the region P', there is a set of extremals

$$y = \phi[x, \xi, \beta(\xi), \beta'(\xi)] = \psi(x, \xi), \tag{9}$$

each one of which is tangent to the curve B at the point $[\xi, \beta(\xi)]$, and such that ψ, ψ', ψ'' are of class C' for

$$x_c \leqq \xi \leqq x_d + \epsilon, \qquad \xi \leqq x \leqq \xi + \rho, \tag{10}$$

provided that ϵ, ρ are properly chosen. The derivative $\partial\psi/\partial\xi$ is a solution of the linear differential equation

$$u'' = ue_y(x, \psi, \psi') + u'e_{y'}(x, \psi, \psi'), \tag{11}$$

as is easily seen by substituting the expression (9) in equation (4) and differentiating for ξ. From the identities

$$\beta(\xi) = \psi(\xi, \xi), \qquad \beta'(\xi) = \psi'(\xi, \xi),$$

it appears that

$$\begin{aligned} \left.\frac{\partial\psi}{\partial\xi}\right|^{x=\xi} &= \beta'(\xi) - \psi'(\xi, \xi) = 0, \\ \left.\frac{\partial\psi'}{\partial\xi}\right|^{x=\xi} &= \beta''(\xi) - \psi''(\xi, \xi) < 0, \end{aligned} \tag{12}$$

the latter on account of the condition IV'. According to a theorem of PICARD, the positive constant ρ can be so restricted independently of ξ that the only integral of equation (11) which vanishes twice on the interval $[\xi, \xi + \rho]$ is $u \equiv 0$.† Equations (12) show therefore that $\partial\psi/\partial\xi$ is not identically zero, but less than zero when $\xi < x \leqq \xi + \rho$.

The field F_B is the interior of the region bounded by the curves in the xy plane which correspond to the boundaries of the region (10) (see Fig. 1). From the properties of the derivative $\partial\psi/\partial\xi$ it can be shown as before that these curves do not intersect, and that the region which they bound is simply covered by the extremals of the set (9). The single-valued function

$$\xi = \xi(x, y)$$

which defines the extremal through a given point, is continuous over the whole

† PICARD, *Traite d'Analyse*, vol. 3, p. 94 ff. If α and β are the maxima of $|e_y|$ and $|e_{y'}|$ in the region (10), then ρ may be any positive quantity satisfying

$$\alpha\frac{\rho^2}{8} + \beta\frac{\rho}{2} < 1,$$

since equation (11) is linear (loc. cit., pp. 97, 100).

region F_B, and has continuous first partial derivatives everywhere except on the curve B. These results follow easily from the theorems on implicit functions when (x, y) is not on the curve B, since the function $\xi(x, y)$ is the solution of

$$y = \psi(x, \xi). \tag{13}$$

As (x, y) approaches a point (x_1, y_1) on the curve B,

$$\lim \xi(x, y) = x_1,$$

as is readily proved by supposing some other limit, and showing that the corresponding extremal cannot pass through (x_1, y_1). The partial derivatives are in general infinite, for

$$0 = \psi_x + \psi_\xi \xi_x,$$

$$1 = \psi_\xi \xi_y,$$

and ψ_ξ vanishes along B.

The first existence theorem of § 1 with ξ_0, η_0, η_0' replaced by x_d, y_d, y_d' shows that the extremals of the set (9) for which

$$x_d - \epsilon \leqq \xi \leqq x_d + \epsilon, \tag{14}$$

can be extended over the interval

$$\xi + \rho \leqq x \leqq x_b, \tag{15}$$

so as to form a field about the curve E_2. The derivative $\partial\psi/\partial\xi$ cannot vanish for any values of x, ξ satisfying (14) and (15), if ϵ is suffificiently small, because from (12) it vanishes at d, and from II′ nowhere else on E_2. The curves corresponding to the boundary of the region defined by (14) and (15), bound a field F_2 in the xy-plane about the extremal E_2, similar to F_1 about E_1.

The three regions F_1, F_B, F_2 form a simply connected plane region F. For part of the boundary of F_B is in F_1, and F_B and F_2 have a strip of boundary in common.

The result is then that *in the neighborhood of the curve E a simply connected plane region F can be found such that each point of F can be joined with the point a' by a curve C consisting either of an extremal which does not intersect E, or else of an arc of E plus an arc of an extremal which is tangent to E* (see Fig. 1). *There is only one such curve C connecting a' with any point of F.*

§ 4. *The Weierstrass construction in the field F.*

With the results of the last section in mind it can be shown by a method of WEIERSTRASS* that the curve E minimizes the integral I with respect to all

*See BOLZA, loc. cit., pp. 84–91 ; also OSGOOD, loc. cit., p. 115.

variations V of class D' which lie in F, join a and b, and do not cross the curve B. This is effected by means of a function W defined as follows: Let $p(\bar{x}, \bar{y})$ be a point which is free to vary along V, then

$$W(\bar{x}) = I_C(a'p) + I_V(pb),$$

where C is the curve described in § 3 joining the point a' with p, $I_C(a'p)$ represents the integral I taken along the curve C from a' to p, and $I_V(pb)$ has an

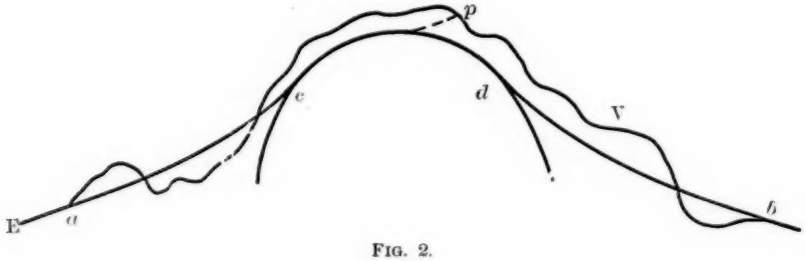

FIG. 2.

analogous meaning (see Fig. 2) with respect to the variation V. The expression whose sign is to be determined is

$$\Delta I = I_V(ab) - I_E(ab) = W(x_a) - W(x_b),$$

and the sign consequently depends upon the behavior of the function $W(\bar{x})$ on the interval $[x_a, x_b]$.

The explicit expression for the function $W(\bar{x})$ is different according as the point p lies in the field F_1 or not. It will be sufficient if its properties are discussed for a point in one of the other fields F_B or F_2. In this case

$$W(\bar{x}) = I_{E_1}(a'c) + \int_{x_c}^{\xi} f(x, \beta, \beta')dx + \int_{\xi}^{\bar{x}} f(x, \psi, \psi')dx + \int_{\bar{x}}^{x_b} f(x, g, g')dx,$$

where $y = g(x)$ is the equation of V, and the function ξ in the limit of the integral and in ψ has the arguments $\bar{x}, g(\bar{x})$ instead of x and y (see the figure). It follows at once from the properties of the function ξ that W is a continuous function of $\bar{x}$. When p is interior to one of the fields F_B or F_2, the derivative of W is

$$\frac{dW}{d\bar{x}} = [f(x, \beta, \beta') - f(x, \psi, \psi')]^{x=\xi}\frac{d\xi}{d\bar{x}} + \frac{d\xi}{d\bar{x}}\int_{\xi}^{\bar{x}}\left\{f_{\psi}\frac{\partial\psi}{\partial\xi} + f_{\psi'}\frac{\partial\psi'}{\partial\xi}\right\}dx + [f(x, \psi, \psi') - f(x, g, g')]^{x=\bar{x}}.$$

Since B and the extremals (9) are tangent at $x = \xi$, this becomes by partial integration and use of the equation (3) which is satisfied by the extremals (9);

$$\frac{dW}{d\bar{x}} = -f(x, g, g') + f(x, \psi, \psi') + \frac{d\xi}{d\bar{x}} \frac{\partial \psi}{\partial \xi} f_{\psi'}(x, \psi, \psi') \Big|^{x=\bar{x}}.$$

But

$$g(x) \equiv \psi\big(x, \xi[x, g(x)]\big),$$

since $\xi(x, y)$ is the solution of equation (13), and therefore

$$g' - \psi' = \frac{\partial \psi}{\partial \xi} \frac{d\xi}{dx}, \tag{16}$$

$$\frac{dW}{d\bar{x}} = -E(x, y, \psi, \psi', g, g') \Big|^{x=\bar{x}},$$

where

$$E = f(x, g, g') - f(x, \psi, \psi') - (g' - \psi') f_{\psi'}(x, \psi, \psi').$$

The function E is always positive or zero and vanishes only when $g' = \psi'$. For by TAYLOR's formula

$$E = \tfrac{1}{2}(g' - \psi')^2 f_{\psi'\psi'}[x, \psi, \psi' + \theta(g' - \psi')],$$

and for a regular problem $f_{\psi'\psi'}$ is always > 0.

For a point p in the field F_1, the derivative of W is the same except that ψ must be replaced by ψ_1. The only points remaining to be considered are points where the curve V coincides with the part $\bar{E}_1$ of E_1 which is common to F_1 and F_B, or with the boundary B. At an interior point of an interval common with one of these curves, it follows readily geometrically that W has a derivative which is zero, for as $(\bar{x}, \bar{y})$ varies along the interval W remains constant. At the endpoints of such an interval, however, the derivative does not in general exist.

To avoid unnecessary complication it will be assumed that V has only a finite number of intervals in common with E_1 or B. Then on the interval $[x_a, x_b]$ the function W is continuous and has a derivative continuous and $\leqq 0$ except in a finite number of points. The expression ΔI will therefore be positive unless $dW/d\bar{x}$ is identically zero in $[x_a, x_b]$. This could happen, however, only if g' were equal to ψ' or ψ_1' at every point of V. In that case V would be of class C' since, for example, in F_B or F_2 the equation

$$g'(x) = \psi'[x, \xi(x), g(x)] \tag{17}$$

implies the continuity of g'. Furthermore V would be an extremal whenever it did not coincide with E. For from (16) ξ must be constant along such an arc.

The only curve in the field with these properties through the point b is E itself, with which therefore V would have to coincide.

The result is then that *the curve E gives to the integral I a smaller value than that given by any other curve of class D' in the field F joining the points a and b.*

A similar theorem holds true when the curve E consists of more than three pieces. Arcs of extremals will then alternate with arcs of the boundary of the region R, and a field can be constructed in the neighborhood of each in the manner described above. The plane region covered by the totality of these fields has the properties of the field F used in the simpler case.

PART II. THE PROBLEM IN PARAMETER REPRESENTATION.

§ 1. *The necessary condition.*

The function F in the integral (2) is assumed to have the following properties in a region of points (x, y, x', y') including in its interior

$$R': \quad \begin{cases} (x, y) \text{ in a given region } R \text{ of the } xy \text{ plane,} \\ (x', y') \text{ any values not both zero:} \end{cases}$$

1) It is of class C''';

2) For all values of κ greater than zero,

$$F(x, y, \kappa x', \kappa y') = \kappa F(x, y, x', y');$$

3) $$F_1 = \frac{1}{y'^2} F_{x'x'} = -\frac{1}{x'y'} F_{x'y'} = \frac{1}{x'^2} F_{y'y'} > 0.$$ *

The region R in which all the curves considered are supposed to lie has a curve of class D^{iv} as its boundary. As before a curve E will be considered consisting of three pieces E_1, B, E_2. The method used in constructing the field makes it necessary to assume that these pieces are of class C^{iv}.

The necessary and the sufficient conditions that E shall minimize the integral read the same as before except that the Euler expression must be replaced by

$$G = F_{xy'} - F_{x'y} + F_1(x'y'' - x''y').$$ †

* For these assumptions on F see BOLZA, loc. cit., p. 117. The function F_1 plays a role analogous to that of $f_{y'y'}$ in the previous case.

† It has been tacitly assumed that the parameter representation on the boundary of R has been so chosen that the interior of R lies to the left as one faces the direction in which the parameter increases. See BOLZA, loc. cit., pp. 123, 148.

§ 2. *An auxiliary theorem.*

The existence of a field about one of the pieces of E follows from the results in the preceding sections provided that the arc of the piece in question and the distances along its normals are taken as new coördinates. The use of these coördinates in a region adjoining the curve is justified by the following theorem: *The normals to a curve of class* $C^{(i)}$ $(i > 1)$ *simply cover a portion of the plane surrounding the curve. In this plane region the arc of the given curve and the distances along the normals can be taken as new coördinates and the transformation so defined will be of class* $C^{(i-1)}$.*

Let the equations of the curve be

$$x = g(s), \qquad y = h(s),$$

where s is the length of the curve measured from some initial point, and let s_0, s_1 correspond to the ends of the arc under consideration. Suppose furthermore that the curve has no singular points on the arc $[s_0, s_1]$, i. e., $g'^2 + h'^2 \neq 0$. The normal at any point has the equations

$$x = g - h't, \qquad y = h + g't, \tag{18}$$

with t as the parameter. It is possible to take a positive constant κ so small that any two pairs of values (s, t) which satisfy

$$s_0 \leqq s \leqq s_1, \qquad |t| \leqq \kappa \tag{19}$$

will define different xy points. For suppose that this were not so. Then for each term of an arbitrarily chosen series of positive constants $\kappa_i (i = 1, 2, \cdots)$ with limit zero, there would exist two points, $P_i(s_i, t_i)$ and $P_i'(s_i', t_i')$, lying in the region (19) defined by κ_i, and such that the corresponding points (x_i, y_i) and (x_i', y_i') defined by equations (18) would be identical. The points P_i, P_i' approach two limiting points $(s, 0)$ and $(s', 0)$ as i increases, and these must also define identical xy points. But this is impossible unless $s = s'$, because the curve is supposed not to intersect itself. The two series of points have therefore the same limit point $(s, 0)$. This leads, however, to a contradiction. For the Jacobian of equations (18) with respect to s and t is

$$(g'^2 + h'^2) \neq 0 \tag{20}$$

at any point $(s, 0)$, and by the theorems on implicit functions there is always a region of (s, t) points surrounding $(s, 0)$ such that any two points of the region define different xy points by means of the equations (8).

* I. e., the functions defining the transformation and their inverses will be of class $C^{(i-1)}$. The method used in the proof of this theorem is similar to the one used by BOLZA in the construction of a field, loc. cit., § 34.

Suppose then a κ selected so that two different points of the region (19) define different xy points, and so that the determinant of equations (18),

$$\begin{vmatrix} x_s & y_s \\ x_t & y_t \end{vmatrix},$$

is different from zero in the whole region. The four curve pieces defined by the points (s, t) on the boundary of (19) form a continuous non-intersecting curve in the xy-plane. By a theorem of SCHOENFLIES * all the xy points corresponding to the region (19) lie in the interior $\bar{R}$ of this curve or on the curve itself; and conversely, to each point of $\bar{R}$ there corresponds a point (s, t) of (19). The equations (18) have therefore single valued solutions s and t in terms of x and y over the region $\bar{R}$, and by the theorems on implicit functions these solutions must be of class $C^{(i-1)}$.

§ 3. *Construction of the field.*

By means of the theorem just proved the problem of constructing a field adjoining one of the pieces of E can be transformed to the st-plane with s as the independent variable. The integral of the transformed problem is

$$I = \int f\left(s, t, \frac{dt}{ds}\right) ds,$$

where

$$f = F\left(x(s, t), y(s, t), x_s + x_t \frac{dt}{ds}, y_s + y_t \frac{dt}{ds}\right).$$

The relations between the Euler equations and the functions $f_{t't'}$ and F_1 are readily calculated and found to be

$$f_t - \frac{d}{ds} f_{t'} = \begin{vmatrix} x_s & y_s \\ x_t & y_t \end{vmatrix} G,$$

$$f_{t't'} = \begin{vmatrix} x_s & y_s \\ x_t & y_t \end{vmatrix}^2 F_1,$$

and from (20) the determinant is positive. Consider for example the extremal E_1. It can be extended slightly at each end of the arc ac, and a region $\bar{R}_1$ which is pictured in a one to one way on a region S_1 in the st-plane, can be constructed as in the preceding section about the extended arc. It is seen that the initial assumptions and conditions I, III, IV′ of § 1 for the function f follow from those of part II, § 1, for the function F. The region in

* See OSGOOD, *Über einen Satz des Herrn Schoenflies aus der Theorie der Functionen zweier reellen Veränderlichen*, Göttinger Nachrichten, 1900.

which they hold consists of all points $(s, t, t' = dt/ds)$ for which (s, t) is in S_1 and $|dt/ds|$ bounded by any arbitrarily chosen positive constant L.

The Jacobi condition II′ is also satisfied. For if

$$x = \mu(s, a), \qquad y = \nu(s, a)$$

is any set of extremals containing E_1 in the xy-plane and corresponding to the set of extremals

$$t = \phi(s, a) \tag{21}$$

in the st-plane, the functional determinant is

$$\begin{vmatrix} \mu_s & \mu_a \\ \nu_s & \nu_a \end{vmatrix} = \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix} \phi_a .$$

The determinant on the left cannot vanish twice on the curve E_1, for its zeros determine conjugate points.* Consequently for any set (21) which contains the transform of E_1, the derivative ϕ_a cannot vanish twice on the segment of the s-axis which corresponds to the arc ac, and this segment in the st-plane cannot contain conjugate points.

It follows that in the st-plane a field can be constructed about the transform of E_1 with a set of extremals

$$t = \phi_1(s_1, \sigma_1), \qquad s_a - \delta \leqq s \leqq s_c + \delta, \qquad |\sigma_1| \leqq \rho,$$

where s_a and s_c are the lengths of arcs of the curve E measured from some fixed point to a and c. In the xy-plane these take the form

$$x = \mu_1(s_1, \sigma_1), \qquad y = \nu_1(s_1, \sigma_1) \tag{22}$$

and on account of the character of the transformation (18) simply cover as before a region F_1 surrounding E_1. Similarly a set of extremals exists in the st-plane

$$t = \psi(s, \sigma), \qquad s_c \leqq \sigma \leqq s_d + \epsilon, \qquad \sigma \leqq s \leqq \sigma + \rho,$$

tangent to the s axis at $s = \sigma$, which in the xy-plane have the form

$$x = \mu(s, \sigma), \qquad y = \nu(s, \sigma), \tag{23}$$

and sweep out a field F_B adjoining the boundary B. By lengthening the arc db of E_2 and transforming again to the st-plane, a set of extremals

$$x = \mu_2(s_2, \sigma_2), \qquad y = \nu_2(s_2, \sigma_2) \tag{24}$$

* See BOLZA, loc. cit., p. 136.

can be found, which are defined for

$$\sigma_2 \leqq s_2 \leqq s_b + \rho, \qquad s_d - \epsilon \leqq \sigma_2 \leqq s_d + \epsilon,$$

and are tangent to the curve B for $s_2 = \sigma_2$. These extremals coincide with those of the set (23) in the vicinity of the curve B, and simply cover a field about E_2.

The regions F_1, F_B, F_2 form a field F about the curve E. Through every point of F not on the curve B there passes one extremal of the field. Each point of the field is joined to the point a' (see Fig. 2) *by one and but one curve of class C' which consists of a finite number of arcs of B and of extremals of the field tangent to B.* In forming the field the part of F_1 which overlaps F_B is to be neglected. The boundary B_1 between the two will then be part of the extremal E_1. Similarly F_B and F_2 can be taken to have part of the curve (24) corresponding to $\sigma_2 = s_d - \epsilon$ as boundary B_2. In the field F_1 the values s_1 and σ_1 defining the extremal through a point (x, y) are single valued functions of x and y, and of class C' since they solve equations (22). The functions s, σ and s_2, σ_2 have similar properties in F_B and F_2, except on the curve B. For points on B they are continuous but the derivatives are not finite.

§ 4. *The Weierstrass construction.*

If V is a curve of class D' in the field F, with the equations

$$x = \bar{x}(\tau), \qquad y = \bar{y}(\tau), \tag{25}$$

and joining $a(\tau = \tau_0)$ with $b(\tau = \tau_1)$, then a function W can be defined geometrically as before along V. In the present case, however, W is a function of τ instead of $\bar{x}$. At any point of V which is not the endpoint of an interval coincident with B_1, B_2 or B, W has the derivative

$$\frac{dW}{d\tau} = -E(x, y, \mu', \nu', \bar{x}', \bar{y}'),$$

where primes denote differentiation with respect to s or τ, and

$$E = F(x, y, \bar{x}', \bar{y}') - \bar{x}' F_{x'}(x, y, \mu', \nu') - \bar{y}' F_{y'}(x, y, \mu', \nu').$$

But, according to WEIERSTRASS,*

$$E = (1 - \cos l) \cdot F_1(x, y, \cos(\lambda + l_1), \sin(\lambda + l_1)) \cdot \sqrt{\bar{x}'^2 + \bar{y}'^2},$$

where

* BOLZA, loc. cit., p. 140.

$$\cos\lambda = \frac{\mu'}{\sqrt{\mu'^2+\nu'^2}}, \qquad \sin\lambda = \frac{\nu'}{\sqrt{\mu'^2+\nu'^2}},$$

$$\cos(\lambda+l) = \frac{\bar{x}'}{\sqrt{\bar{x}'^2+\bar{y}'^2}}, \qquad \sin(\lambda+l) = \frac{\bar{y}'}{\sqrt{\bar{x}'^2+\bar{y}'^2}},$$

and l_1 is a value between 0 and l. Consequently E is always positive unless $l \equiv 0$ along V. If therefore V has only a finite number of intervals in common with B_1, B_2 and B, and if $l \not\equiv 0$ on V, then

$$\Delta I = W(\tau_0) - W(\tau_1)$$

is surely positive.

If $l \equiv 0$ along V, then V and E must coincide throughout. For the equations

$$\frac{\bar{x}'}{\sqrt{x'^2+y'^2}} = \frac{\mu'}{\sqrt{\mu'^2+\nu'^2}}, \qquad \frac{\bar{y}'}{\sqrt{x'^2+y'^2}} = \frac{\nu'}{\sqrt{\mu'^2+\nu'^2}}, \tag{26}$$

show that V is of class C'. In the vicinity of any point of V not on E it must coincide with the extremal of the field through that point. For from equations (23), for example, when x and y are replaced by the expressions (25),

$$\bar{x}' = \mu' \frac{ds}{d\tau} + \mu_\sigma \frac{d\sigma}{d\tau},$$

$$\bar{y}' = \nu' \frac{ds}{d\tau} + \mu_\sigma \frac{d\sigma}{d\tau}.$$

But from (26),

$$0 = \mu' \left(\frac{ds}{d\tau} - \frac{\sqrt{\bar{x}'^2+\bar{y}'^2}}{\sqrt{\mu'^2+\nu'^2}} \right) + \mu_\sigma \frac{d\sigma}{d\tau},$$

$$0 = \nu' \left(\frac{ds}{d\tau} - \frac{\sqrt{\bar{x}'^2+\bar{y}'^2}}{\sqrt{\mu'^2+\nu'^2}} \right) + \nu_\sigma \frac{d\sigma}{d\tau},$$

and these can only hold if σ is constant as τ varies, and therefore V an extremal. But E is the only curve of the field with the properties just found and passing through the point b. It follows then that V must be the curve E itself.

It has been proved that *for the problem in parameter representation the curve E minimizes the integral I with respect to all variations of class D' in the field F.*

The derivation of the necessary and of the sufficient conditions for the problem in parameter representation is of especial interest because it has been proved by BOLZA,* following a method suggested by HILBERT,† that there is certainly one curve in the region R joining the points a and b, which gives to the integral I its minimum value, provided that proper assumptions are made upon the function F. In the simpler case when I has the form (1) this is not always true. The necessary conditions given above characterize this solution of the problem, and as has been seen, when altered slightly, they are also sufficient for a minimum in a certain neighborhood of the curve.

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* Loc. cit., chap. vii, pp. 245–263.

† *Lectures on the Calculus of Variations*, Göttingen, Summer, 1900.

SUR LES OPÉRATIONS LINÉAIRES*

PAR

MAURICE FRÉCHET

I. Rappelons d'abord quelques définitions. Nous dirons qu'une *opération* est définie si l'on fait correspondre un nombre réel déterminé et fini U_f à toute fonction $f(x)$ réelle et continue entre deux nombres fixes a, b. Nous appellerons avec M. HADAMARD *opération linéaire* toute opération qui jouit des deux propriétés suivantes:

1° elle est *distributive*, c'est à dire que si f_1 et f_2 sont deux fonctions continues entre a et b, on a toujours:

$$U_{f_1+f_2} = U_{f_1} + U_{f_2}. \tag{1}$$

2° elle est *continue*, c'est à dire que U_{f_1} tend vers U_{f_2} lorsque la fonction f_1 tend *uniformément* vers f_2 entre a et b.

M. HADAMARD a démontré † en se basant sur l'étude de l'expression:

$$\lim_{\mu=\infty} \int_a^b f(u)\, \mu e^{-\mu^2(u-x)^2}\, du, \tag{2}$$

que toute opération linéaire peut être représentée sous la forme:

$$U_f = \lim_{n=\infty} \int_a^b f(x)\, K_n(x)\, dx, \tag{3}$$

où $K_n(x)$ est une fonction *continue* de x entre a et b.

Je me propose de donner ici une nouvelle démonstration de cet important théorème, démonstration qui me permettra d'établir simultanément un développement en série très général de U_f. Je présenterai ensuite quelques remarques sur l'expression (3).

Développements en séries d'opérations linéaires canoniques.

II. Il est assez naturel de chercher à établir pour les opérations linéaires un développement analogue à celui de Taylor pour les fonctions ordinaires. Il s'agirait de trouver une suite d'opérations linéaires simples: $U_f^{(0)}, \cdots, U_f^{(n)}, \cdots$ choisies une fois pour toutes et telles que toute opération linéaire U_f puisse s'écrire sous la forme:

* Presented to the Society at the St. Louis meeting, September 17, 1904. Received for publication June 24, 1904.

† Comptes Rendus: *Sur les opérations fonctionnelles*, 9 Février, 1903.

$$U_f = u_0 U_f^{(0)} + u_1 U_f^{(1)} + \cdots + u_n U_f^{(n)} + \cdots, \tag{4}$$

les constantes $u_0, u_1, \cdots$ variant seules avec l'opération U_f.

M. PINCHERLE a déja montré que l'on pouvait écrire:

$$U_f = u_0 f(c) + u_1 f'_c + \cdots + u_n f^{(n)}_c + \cdots, \tag{5}$$

c étant un nombre fixe compris entre a et b, en prenant:

$$u_0 = U_1,\ u_1 = U_{(x-c)}, \cdots,\ u_n = \frac{1}{n!} U_{(x-c)^n}, \cdots.$$

Mais un tel développement ne peut *s'écrire* que si $f(x)$ est dérivable indéfiniment en $x = c$ et encore* cela ne suffit pas pour assurer la convergence du développement vers U_f.

Représentation par une série convergente de toute opération linéaire portant sur une fonction satisfaisant aux conditions de Dirichlet.

III. Je vais montrer d'abord qu'on peut obtenir un développement de la forme (4) ayant un champ de validité* beaucoup plus étendu que le développement (5) puisqu'il n'exige même pas que $f(x)$ soit dérivable.

Pour cela, observons que l'on peut toujours supposer avoir remplacé l'intervalle (a, b) par l'intervalle $(0, \pi)$. Ne considérons pour le moment que les fonctions $f(x)$ continues dans $(0, \pi)$ et n'ayant qu'un nombre limité de maxima et de minima dans cet intervalle. Si l'on développe en série de Fourier la fonction (satisfaisant aux conditions de Dirichlet) qui est égale à $f(x)$ entre 0 et π et à $f(-x)$ entre 0 et $-\pi$, on aura entre 0 et π:

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + \cdots, \tag{6}$$

avec

$$a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx, \qquad a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx \quad (n = 1, 2, \cdots). \tag{7}$$

IV. Puisque la série (6) converge *uniformément* dans $(0, \pi)$, on peut lui appliquer *terme à terme* toute opération linéaire U_f et l'on aura:

$$U_f = \frac{\pi}{2}(2a_0 u_0 + a_1 u_1 + \cdots + a_n u_n + \cdots), \tag{8}$$

en posant cette fois:

$$u_0 = \frac{1}{\pi} U_1,\ u_1 = \frac{2}{\pi} U_{\cos x}, \cdots,\ u_n = \frac{2}{\pi} U_{\cos nx}, \cdots. \tag{9}$$

Ce développement est bien de la forme (4), car les formules (7) montrent que les

* D'ailleurs, M. PINCHERLE ne s'occupant que d'opérations portant sur des fonctions analytiques, la remarque a, dans ce cas, beaucoup moins de portée. Voir *Le operazioni distributive* par S. PINCHERLE et U. AMALDI, 1901, Bologne.

quantités $a_0, a_1, \cdots$, peuvent être considérées comme des opérations linéaires déterminées une fois pour toutes:

$$(10) \qquad U_f^{(0)} = \int_0^\pi f(x)\,dx, \ \cdots, \ U_f^{(n)} = \int_0^\pi f(x)\cos nx\,dx, \ \cdots.$$

Ainsi, *nous sommes parvenus à un développement de la forme* (4), moyennant les formules (9) et (10), *développement qui est valable pour toutes les fonctions continues ayant un nombre limité de maxima et de minima entre* 0 *et* π.

Représentation par une série simplement indéterminée de toute opération linéaire portant sur une fonction continue quelconque.

V. On peut même s'arranger pour supprimer la dernière condition en généralisant convenablement la notion de limite. Pour cela, nous appellerons, avec M. CESARO, *limite généralisée* d'une suite de nombres $u_0, u_1, u_2, \cdots$, la limite, si elle existe, de la suite,

$$\frac{u_0+u_1}{1}, \ \frac{u_0+u_1+u_2}{2}, \ \cdots, \ \frac{u_0+u_1+\cdots+u_n}{n}, \ \cdots.$$

Nous dirons qu'une série est *simplement indéterminée* si la somme de ses n premiers termes a une limite généralisée qui sera par définition la *somme généralisée* de la série. Pour une telle série:

$$v_0, \ v_1, \ \cdots, \ v_n, \ \cdots,$$

de somme généralisée σ, nous écrirons:

$$\sigma = v_0 + v_1 + \cdots + v_n + \cdots,$$

ou:

$$\sigma = \lim_{n=\infty} \text{gén } S_n,$$

en posant

$$S_n = v_0 + \cdots + v_n,$$

ou:

$$\sigma = \lim_{n=\infty} \sigma_n$$

avec:

$$\sigma_n = \frac{S_0 + \cdots + S_n}{n} = \frac{nv_0 + (n-1)v_1 + \cdots + v_n}{n}.$$

On sait, d'ailleurs, que si une série est convergente, elle a une somme généralisée égale à sa somme au sens ordinaire.

Or, M. FEJER a démontré* que toute fonction périodique continue peut être représentée par la somme généralisée de son développement de FOURIER, avec une convergence uniforme.

*Mathematische Annalen, Band 58 (1903), Heft 1, 2.

Par suite, *le développement* (8) *est valable pour toute fonction* $f(x)$ *continue entre* 0 *et* π *pourvu qu'on remplace la somme de la serie par sa somme généralisée.*

On aura ainsi:

$$(6)' \qquad f(x) = a_0 + a_1 \cos x + \cdots + a_n \cos nx + \cdots,$$

$$(4)' \qquad U_f = u_0 U_f^{(0)} + \cdots + u_n U_f^{(n)} + \cdots,$$

avec les formules (7), (9) et (10).

Le théorème de M. Hadamard.

VI. Quelle que soit la fonction continue $f(x)$, les égalités (6)′ et (4)′ pourront s'écrire:

$$(11) \qquad f(x) = \lim_{n=\infty} \int_0^\pi f(y)\sigma_n(xy)\,dy,$$

$$(12) \qquad U_f = \lim_{n=\infty} \int_0^\pi f(y)\,K_n(y)\,dy,$$

en posant:

$$(13) \quad \sigma_n(x, y) = \frac{2}{\pi}\left(\frac{\frac{1}{2}n + (n-1)\cos y \cos x + \cdots + \cos ny \cos nx}{n}\right),$$

$$(14) \qquad K_n(y) = \frac{nu_0 + (n-1)u_1 \cos y + \cdots + u_n \cos ny}{n}.$$

La fonction $K_n(y)$ est évidemment continue. Par suite, *le théorème de M. Hadamard résulte des formules* (12) et (14). Car il suffit d'y effectuer la substitution: $y = a + (b-a)z/\pi$ pour revenir au cas d'un intervalle quelconque (a, b).

Remarques sur les modes de représentation précédents.

VII. Les fonctions $K_n(y)$ sont bien déterminées par l'opération U_f, car on a:

$$(15) \qquad K_n(y) = U_{\sigma_n(x, y)},$$

en considérant dans $\sigma_n(x, y)$ la variable y comme un paramètre arbitraire.

M. HADAMARD (loc. cit.) a fait observer que la condition nécessaire et suffisante pour que l'opération U_f se présente sous la forme:

$$(16) \qquad U_f = \int_0^\pi f(y)\,H(y)\,dy$$

où $H(y)$ est une fonction continue, est que les fonctions K_n convergent uniformément vers une fonction limite. Je ferai remarquer que la méthode actuelle permet de formuler cette condition d'une manière assez simple si l'on part de l'expression (4)′. En effet, les fonctions $K_n(y)$ représentent la quantité dont on

doit chercher la limite ordinaire pour obtenir, lorsqu'elle existe, la somme généralisée de la série de Fourier:

$$K(y): \quad u_0 + u_1 \cos y + \cdots + u_n \cos ny + \cdots. \tag{17}$$

Par conséquent, *la condition nécessaire et suffisante pour qu'une opération linéaire U_f puisse être mise sous la forme intégrale (16) est que les coefficients $u_0, u_1, \cdots$ de son développement en série: (8)', soient les coefficients du développement en serie de Fourier (17) d'une fonction continue.*

VIII. Le théorème de M. HADAMARD conduit naturellement à étudier le problème suivant: puisque toute opération linéaire peut s'exprimer sous la forme:

$$U_f = \lim_{n=\infty} \int_0^\pi f(y) H_n(y)\, dy, \tag{18}$$

$H_n(y)$ étant une fonction continue, à quelles conditions le second membre pourra-t-il définir une opération linéaire si l'on se donne a priori une suite quelconque de fonctions continues $H_n(y)$?

Il est évidemment suffisant que les fonctions H_n tendent uniformément vers une fonction limite nécessairement continue $H(y)$. Mais cela n'est pas nécessaire; *ce n'est même pas nécessaire pour que l'expression:*

$$V_f^{(n)} = \int_0^\pi f(y) H_n(y)\, dy, \tag{19}$$

ait pour limite une opération linéaire de la forme:

$$\int_0^\pi f(y) H(y)\, dy, \tag{16}$$

où $H(y)$ est continue.

En effet, nous allons montrer que l'on peut construire une suite de fonctions continues $H_n(y)$ ayant pour limite une fonction $K(y)$ dont l'ensemble des discontinuités a la puissance du continu et telles cependant que l'opération $V_f^{(n)}$ ait pour limite une opération de la forme (16) où $H(y)$ est continue.

En effet, supposons formée la suite des fonctions H_n de façon que la limite $K(y)$ ne diffère d'une certaine fonction continue $H(y)$ qu'en tous les points d'un ensemble de mesure nulle E, les fonctions H_n et K restant bornées dans leur ensemble. On sait dans ce cas que les fonctions $H_n f$ auront pour limite une fonction Kf mesurable * et que l'intégrale $\int_0^\pi H_n f\, dy$ aura pour limite † l'intégrale au sens de M. LEBESGUE $\int_0^\pi Kf\, dy$. Or, on aura: ‡

* *Leçons sur l'intégration* par H. LEBESGUE, p. 111, Gauthier Villars, Paris, 1904.

† Loc. cit., p. 114.

‡ Loc. cit., p. 116.

$$\int_0^\pi Kf\,dy - \int_0^\pi Hf\,dy = \int_E (K-H)f\,dy = 0,$$

puisque E est de mesure nulle et que la fonction $(K-H)f$ est bornée. On a donc bien:

$$\lim_{n=\infty} \int_0^\pi H_n f\,dy = \int_0^\pi Hf\,dy,$$

H étant continue.

Pour former la suite des fonctions H_n, il suffit de prendre pour E, un ensemble parfait, non dense et de mesure nulle comme celui qui a été défini par M. CANTOR.* Un tel ensemble, qui a la puissance du continu, s'obtient en enlevant du segment $(0, \pi)$ tous les points intérieurs au sens étroit à certains intervalles $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n), \cdots$.

Posons alors:

$$a_p^{(n)} = a_p + \frac{b_p - a_p}{n-p+3}, \qquad b_p^{(n)} = b_p - \frac{b_p - a_p}{n-p+3} \qquad (n \geqq p).$$

L'intervalle $(a_p^{(n)}, b_p^{(n)})$ est complétement intérieur à l'intervalle (a_p, b_p) et les points $a_p^{(n)}$, $b_p^{(n)}$ tendent respectivement vers a_p, b_p lorsque p restant fixe, n croit indéfiniment.

Il est alors facile de voir qu'on arrivera au but indiqué en prenant $H_n(y) = H(y)$ dans les intervalles: $(a_1^{(n)}, b_1^{(n)}), \cdots, (a_n^{(n)}, b_n^{(n)})$, puis $H_n(y) = R(y)$ en dehors des intervalles $(a_1, b_1), \cdots, (a_n, b_n)$, $R(y)$ étant une fonction quelconque déterminée et bornée dans $(0, \pi)$, enfin en prenant pour $H_n(y)$ une fonction linéaire dans les intervalles restants où on connait les valeurs de H_n aux extrémités. Si $R(y) - H(y)$ ne s'annule en aucun point de E, ce qu'on peut toujours supposer, tous les points de E seront des points de discontinuité de $K(y)$.

IX. Ce qui précède montre en outre l'utilité qu'il y a à ne pas rejeter comme trop artificielles des opérations qui seraient définies par une expression telle que:

$$\int_0^\pi f(y) K(y)\,dy. \tag{20}$$

où K serait une fonction non continue, même pas intégrable au sens de RIEMANN, mais intégrable au sens plus large de M. LEBESGUE (comme la fonction qui est égale à 1 en tous les points d'abscisses irrationnelles et à 2 en tous les autres points). Car on s'exposerait à rejeter en même temps (comme dans l'exemple précédent) des opérations linéaires très simples et très naturelles. Nous sommes donc conduits à poser un problème un peu plus précis que le précédent

* Voir par exemple *Leçons sur les fonctions de variables réelles*, page 12, par É. BOREL, Gauthier Villars, Paris, 1904.

(§ VIII). Dans quel cas l'opération (19) a-t-elle pour limite une opération de la forme (20), où la fonction $K(y)$ est la fonction la plus générale qui puisse donner un sens* à cette expression, c'est-à-dire une fonction mesurable? Comme nous l'avons vu, il suffit pour cela que les fonction H_n soient bornées dans leur ensemble et tendent (de façon quelconque) vers une fonction limite. Mais, ici encore, cela n'est pas nécessaire car toute fonction limite de fonctions continues est de première classe (au sens de M. BAIRE), tandis que $K(y)$ peut être mesurable sans être de première classe.†

X. Quoiqu'il en soit, nous allons montrer (après ces exemples de conditions suffisantes) que l'on peut donner à la première question que nous avons posée une réponse partielle présentant un criterium d'une assez grande simplicité.

Je dis que *si l'expression:*

$$V_f^{(n)} = \int_0^\pi H_n(y) f(y)\, dy,$$

où les H_n sont des fonctions continues quelconques, a pour limite une opération linéaire U_f, quand n croit indéfiniment, les développements de Fourier des fonctions $H_n(y)$ ont un développement limite.

Autrement dit, si l'on écrit, comme on en a le droit:

$$H_n(y) = u_0^{(n)} + u_1^{(n)} \cos y + \cdots + u_p^{(n)} \cos py + \cdots$$

la suite:

$$u_p^{(1)}, u_p^{(2)}, \cdots, u_p^{(n)}, \cdots,$$

a toujours une limite u_p quel que soit p.

Il suffit d'observer que, par hypothèse, l'expression $V_f^{(n)}$ a une limite bien déterminée U_f pour toute fonction $f(y)$ continue entre 0 et π. Le théorème s'obtient alors immédiatement en appliquant cette condition aux fonctions

$$f(y) = \frac{1}{\pi}, \; f(y) = \frac{2\cos y}{\pi}, \; \cdots, f(y) = \frac{2\cos py}{\pi}, \; \cdots.$$

Si le développement limite:

$$u_0 + u_1 \cos y + \cdots + u_n \cos ny + \cdots,$$

constitue une série simplement indéterminée et représente une fonction continue $K(y)$, la condition sera suffisante. Mais dans le cas général, nous n'en savons rien. Je me propose d'étudier cette question dans une seconde note.

PARIS, *Juin* 1904.

* Voir LEBESGUE, loc. cit., page 98.

† Voir LEBESGUE, loc. cit., page 112.

ON GROUPS IN WHICH CERTAIN COMMUTATIVE OPERATIONS ARE CONJUGATE*

BY

H. L. RIETZ

Introduction.

The groups in which every two conjugate operators are commutative have recently been considered by BURNSIDE.† In the first section of the present paper, the converse limitation is imposed on a group of operations. It is assumed that every two commutative operations are conjugate, provided neither is identity,‡ and the groups which are possible under this hypothesis are determined. It results that the group of order 2§ and the symmetric group of order 6 are the only groups which have the property in question.

In sections 2–6, somewhat similar but smaller limitations are imposed on the group. The condition is imposed in sections 2–5 that every two commutative operations of the same order are conjugate, and in section 6 that every two commutative operations (identity excluded) are so related that each of them is conjugate to some power of the other. Some of the chief properties of the groups which are possible under these limitations are derived.

The sections 7–8 deal with problems closely related to the preceding. If it is assumed that a certain simple relation exists between the number λ of complete sets of conjugate operations, and the number n of distinct prime factors in the order of the group, certain commutative operations are conjugate. Much use is made of this fact in showing what groups are possible under the given hypothesis.

The symbol $G \equiv (S_1 = 1), S_2, S_3, \cdots, S_g$ will be used to represent the group of order g under consideration, and $p_1, p_2, \cdots, p_n$ to represent distinct primes in ascending order of magnitude so that $g = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$.

* Presented to the Society (Chicago) December 31, 1903. Received for publication January 18, 1904.

† Proceedings of the London Mathematical Society, vol. 35 (1903), pp. 28–37.

‡ Without making an exception of identity, clearly no groups are possible.

§ The group of order 2 may be regarded as a trivial case.

§ 1. *Groups in which every two commutative operations, apart from the identical operation, are conjugate.*

§ 1. It is assumed, in case

$$S_a S_b = S_b S_a \qquad (a>1,\ b>1), \tag{1}$$

that there is in G an operation S_c such that

$$S_c^{-1} S_a S_c = S_b. \tag{2}$$

Then S_a and S_b must be of the same order, that is, any two commutative operations of G, neither of which is the identical operation, must be of the same order. Since an operation is commutative with all its powers, all the powers of an operation, except identity, must be of the same order. But this is clearly true only of operations of prime order. Hence, all operations in G must be of prime order. Any Sylow subgroup H of order $p_r^{a_r}$ contains at least p_r operations invariant in H. Hence all the operations in H, except identity, belong to the same conjugate set under G. Since the Sylow subgroups are themselves conjugate, all the operations of order p_r are conjugate, and each is invariant in a subgroup of order $p_r^{a_r}$. Hence the operations of G, aside from identity, are distributed into just as many complete sets of conjugates as there are distinct prime factors in g, and $g/p_r^{a_r}$ is the number of operations of G of order p_r.

Hence

$$\sum_{r=1}^{r=n} \frac{g}{p_r^{a_r}} = g - 1. \tag{3}$$

Remembering that p_n is the largest prime number contained in g, let S_n be an operation of order p_n. Since there must be in G operations which transform S_n into its $p_n - 1$ powers of order p_n, let

$$S_t^{-1} S_n S_t = S_n^{\alpha},$$

where α is a primitive root of p_n. Then

$$S_t^{-q} S_n S_t^{q} = S_n^{\alpha^q}.$$

If q is the order of S_t,

$$\alpha^q \equiv 1 \mod p_n,$$

and

$$q = p_n - 1.$$

But p_n and $p_n - 1$ cannot both be prime numbers unless $p_n = 3$ or 2. Hence

$$g = 2^{a_1} 3^{a_2} \qquad (a_1 \geqq 1,\ a_2 \geqq 0). \tag{4}$$

If $a_2 = 0$, we have from (3) and (4)

$$g=2^{\alpha_1}, \qquad \frac{g}{2^{\alpha_1}}=g-1,$$

and so, in this case, the group G would be of order 2. The group of order 2 does satisfy the prescribed condition (1, 2) since it has only one operation besides identity.

If $\alpha_2>0$, we have from (3) and (4)

$$g=2^{\alpha_1}3^{\alpha_2}, \qquad \frac{g}{2^{\alpha_1}}+\frac{g}{3^{\alpha_2}}=g-1,$$

and so

$$3^{\alpha_2}=1+\frac{2}{2^{\alpha_1}-1}, \qquad \alpha_1=1, \alpha_2=1.$$

We thus find that $g=6$. It is easy to verify that the symmetric group of order 6 satisfies the prescribed condition (1, 2), while the cyclic group of order 6 does not. Hence,

The symmetric group of order 6, *and the group of order* 2 *are the only groups in which every two commutative operations, apart from identity, are conjugate.*

It also follows at once, from the method of proof, that

These two groups are the only groups which have both the properties that all their operations are of prime order, and all those of the same order are conjugate.

§§ 2–6. *Groups in which every two commutative operations of the same order are conjugate.*

§ 2. It is now assumed, in case

(1) $$S_a S_b = S_b S_a,$$

S_a and S_b being of the same order, that there is in G an operation S_c such that

(2) $$S_c^{-1} S_a S_c = S_b.$$

From (1),

$$S_a S_b^{-1} = S_b^{-1} S_a,$$

and there must therefore be in G an operation S such that

(3) $$S^{-1} S_b^{-1} S = S_a.$$

Multiplying both members of (3) by S_b, we have

$$S^{-1} S_b^{-1} S S_b = S_a S_b.$$

Hence the product of any two commutative operations of the same order is a commutator. In particular, *the square of every operation is a commutator.*

Therefore, if we represent the commutator subgroup by C, and it does not include all the operations of G, the abelian * quotient group G/C contains only operations of order 2 besides identity. Hence:

A group in which every two commutative operations of the same order are conjugate, is either a perfect† group, or isomorphic to the abelian group of order 2^α of type $(1, 1, 1, \cdots, 1)$.

§ 3. If g is restricted to be a power of a single prime, it is easy to show what groups are possible under the given limitations. Since any group of order p^α (p any prime number) contains invariant operations besides identity, suppose S is such an operation of G. Since every two commutative operations of the same order are conjugate, G can contain no other operations of the same order as S. Hence $S^2 = 1$ and $p = 2$.

With $g = 2^\alpha$, G must either be the cyclic group ‡ or the type defined by

$$P^{2^{\alpha-1}} = 1, \qquad Q^2 = P^{2^{\alpha-2}}, \qquad Q^{-1}PQ = P^{-1},$$

in order to contain a single substitution of order 2. If G is cyclic, obviously $g = 2$. If, however, G has the defining relations just written, there can be only two operations of order $2^{\alpha-1}$ in the subgroups defined by $P^{2^{\alpha-1}} = 1$, since these operations must be conjugate under G. From this it readily follows that $\alpha = 3$, and the defining relations of G are

$$P^4 = 1, \qquad Q^2 = P^2, \qquad Q^{-1}PQ = P^{-1},$$

so that G is the quaternion group. Hence,

The group of order 2 and the quaternion group are the only groups of order p^α (p any prime number) in which every two commutative operations of the same order are conjugate.

§ 4. Suppose that in the group G every two operations of the same order are conjugate, whether commutative or non-commutative. By extending the argument of § 2 it is easy to establish an important property of these groups. These groups are, of course, a special class of the groups we have been considering. Corresponding to any operation S of order 2 in G/C (§ 2), there must be in G an operation S_1 of order 2^{α_1} ($\alpha_1 \geqq 1$), while S_1^2 of order 2^{α_1-1} is contained in C and no operation of order 2^{α_1} can be contained in C if all operations of the same order are conjugate. Corresponding to any operations S' of order 2 in G/C, the group G must contain operations of order 2^{α_1}. Since G/C is abelian, these

* MILLER, Quarterly Journal of Mathematics, vol. 28 (1896), pp. 266-268.

† A group identical with its commutator subgroup is a perfect group. Cf. LIE, *Transformationsgruppen*, vol. 3, p. 679.

‡ BURNSIDE, *Theory of Groups of Finite Order* (1897), p. 75.

operations could not be conjugate to the operations of the same order which correspond to S. Hence S is the only operation in G/C besides the identical operation. It results, therefore, that

A group in which every two operations of the same order are conjugate is a perfect group, or else the commutator subgroup contains just one half the operations of the group.

§ 5. In the article by BURNSIDE referred to in the introduction to this paper, it is shown that in a group in which every two conjugate operations are commutative the commutators are invariants unless the order of the group is a power of 3. Combining this result with the one just stated in § 4 we are enabled to state that

The group of order 2 is the only group in which all operations of the same order are both conjugate and commutative.

§ 6.* *Groups in which every two commutative operations (identity excluded) are so related that each of them is conjugate to some power of the other.*†

If S_a and S_b are any two operations of G such that $S_a S_b = S_b S_a$, we now assume that there occur in G operations S_c and S_c' such that

$$S_c^{-1} S_a S_c = S_b^{\alpha} \tag{1}$$

and

$$S_c'^{-1} S_b S_c' = S_a^{\beta}. \tag{2}$$

From (1) and (2), S_a and S_b are of the same order. Since every two commutative operations (identity excluded) are of the same order, as in § 1, all the operations of the group are of prime order. Consider a Sylow subgroup H_r of order $p_r^{\alpha_r}$. Since any group of order $p_r^{\alpha_r}$ contains invariant operations, it follows that all subgroups of order p_r contained in H_r are conjugate under G. Since the Sylow subgroups are themselves conjugates, it follows that *all subgroups of the same prime order contained in G are conjugate, and that every operation of G is invariant in a Sylow subgroup.*

Let C represent the commutator subgroup of G. Since G/C is abelian and has all its operations of prime order, it must be a group of order $p_r^{\alpha_r}$ of type $(1, 1, \cdots, 1)$. If $\alpha_r > 1$, it is clearly impossible for all the subgroups of order p_r contained in G to be conjugate. Hence

A group in which every two commutative operations, apart from the identity, are each conjugate to some power of the other is a perfect group or else its commutator subgroup is of prime index.

* Added July, 1904.

† The groups of § 8 have this property

§§ 7–8. *On the complete sets of conjugate operations in a group.*

§ 7. In § 1 we found that the groups under consideration have just as many complete sets of conjugate operations, aside from identity, as there are distinct prime factors in the order. Since G contains, by Cauchy's theorem, operations of orders $p_1, p_2, \cdots, p_n$, it cannot have fewer complete sets of conjugate operations, even if we neglect identity, than there are distinct prime factors in g. Hence, if λ is the number of complete sets of conjugates without identity, $\lambda \geqq n$. If $\lambda = n$, every operation must be of prime order, and must be conjugate to every other operation of the same order.

From the reasoning of § 1, we have that *the group of order* 2, *and the symmetric group of order* 6 *are the only groups in which the number of complete sets of conjugate operations* (*without identity*) *is exactly equal to the number of distinct prime factors in the order.*

§ 8. It may be of interest to determine what groups are possible when $\lambda = n + 1$, as the solution of this problem can be made to depend largely upon the fact that certain commutative operations must be conjugate. For $n = 1$, the group of order 3 is the only one with $\lambda = n + 1$. We consider a group G with $n \geqq 2$ and $\lambda = n + 1$. The operations of G, apart from those of one conjugate set, must be of prime order. The order of the operations of this conjugate set cannot contain more than two prime factors, or there would occur in G operations of more than $n + 1$ different orders.

We shall divide the consideration of these groups into three parts:

1°. When G contains an operation of order $p_r p_s$ (p_r and p_s distinct). In this case all operations of the same order are conjugate.

Since an operation of order p_n must be transformed into all its powers except identity, just as in § 1, G must contain operations of order $p_n - 1$. Hence

$$p_n - 1 = 2p_t \qquad (p_t \text{ a prime} \neq 2).$$

If $p_t > 1$, $p_t - 1 = 2$, since G cannot contain operations of two different composite orders. Hence $g = 2^{\alpha_1} 3^{\alpha_2}$, or $2^{\alpha_1} 3^{\alpha_2} 7^{\alpha_3}$. If $g = 2^{\alpha_1} 3^{\alpha_2}$, each operation is clearly transformed into itself by at least 6 operations. In each case to be considered we shall write some inequalities or equations from the distribution of operations into conjugate sets without detailed explanation. Thus, in this case,

$$(A) \qquad \frac{g}{6} + \frac{g}{6} + \frac{g}{6} \geqq g - 1.$$

Similarly, if $g = 2^{\alpha_1} 3^{\alpha_2} 7^{\alpha_3}$

$$(B) \qquad \frac{g}{7} + \frac{g}{6} + \frac{g}{6} + \frac{g}{6} + 1 \geqq g.$$

2°. When G contains an operation of order p_r^2.

By the method employed in 1°, $p_n - 1 = 2^2$ or 2. Hence

$$g = 2^{\alpha_1}3^{\alpha_2} \quad \text{or} \quad g = 2^{\alpha_1}3^{\alpha_2}5^{\alpha_3} \qquad (\alpha_2 \geqq 0).$$

If $g = 2^{\alpha_1}3^{\alpha_2}$, either

$$(C) \qquad \frac{g}{2} + \frac{g}{9} + \frac{g}{9} + 1 \geqq g, \quad \text{or} \quad \frac{g}{2^2} + \frac{g}{2^2} + \frac{g}{3} + 1 \geqq g.$$

If $g = 2^{\alpha_1}5^{\alpha_3}$,

$$(D) \qquad \frac{g}{2^2} + \frac{g}{2^2} + \frac{g}{5} \geqq g - 1.$$

If $g = 2^{\alpha_1}3^{\alpha_2}5^{\alpha_3}$,

$$(E) \qquad \frac{g}{2^{\alpha_1}} + \frac{g}{2^\beta} + \frac{g}{3^{\alpha_2}} + \frac{g}{5^{\alpha_3}} = g - 1 \qquad (\alpha_1 \geqq 2,\ \beta \geqq 2,\ \alpha_2 \geqq 1,\ \alpha_3 \geqq 1).$$

It is easily seen that (A), (B), (C), (D) and (E) cannot be satisfied by the given values of g. There is, therefore, no group with $\lambda = n + 1$ which contains operations of composite order.

3°. When all the operations of G are of prime order. There can be only one prime which is the order of two operations that are not conjugate. If this prime is not the largest prime factor in g, by the reasoning of § 1, the largest prime factor is 3 and

$$g = 2^{\alpha_1}3^{\alpha_2} \qquad (\alpha_1 \geqq 2,\ \alpha_2 \geqq 1).^*$$

Hence,

$$(F) \qquad \frac{g}{2^2} + \frac{g}{2^2} + \frac{g}{3} \geqq g - 1, \ \cdots.$$

But (F) cannot be satisfied by

$$g = 2^{\alpha_1}3^{\alpha_2} \qquad (\alpha_1 \geqq 2,\ \alpha_2 \geqq 1).$$

The prime which is the order of two operations that are not conjugate is then p_n, and from the argument of § 1, $p_{n-1} \not> 3$. Hence,

$$g = 2^{\alpha_1}3^{\alpha_2}p_3^{\alpha_3} \qquad (\alpha_1 \geqq 1,\ \alpha_2 \geqq 0,\ \alpha_3 \geqq 0).$$

We shall first consider the case where $\alpha_3 > 0$. Then every operation must be invariant in a Sylow subgroup. This statement follows at once from the fact that every Sylow subgroup contains invariant operations, except in the case of the one conjugate set of operations of order p_3. An invariant operation P of order p_3 and P^α where α is a primitive root of p_3 must belong to different conjugate sets. Since P is invariant, P^α is also invariant.

If $\alpha_2 = 0$ while $\alpha_3 > 0$,

$$\frac{g}{2^{\alpha_1}} + \frac{g}{p_3^{\alpha_3}} + \frac{g}{p_3^{\alpha_3}} = g - 1,$$

* $\alpha_1 \geqq 2$, for if $\alpha_1 = 1$ the operators of order 2 are conjugates, by Sylow's theorem.

and therefore

$$(G) \qquad p_3^{\alpha_3} + 2^{\alpha_1} + 2^{\alpha_1} = 2^{\alpha_1} p_3^{\alpha_3} - 1 \qquad \text{or} \qquad p_3^{\alpha_3} = 2 + \frac{3}{2^{\alpha_1} - 1}.$$

Hence $\alpha_1 = 1$, $\alpha_3 = 1$, and $g = 10$ since $p_3 > 3$.

If $\alpha_2 > 0$ while $\alpha_3 > 0$,

$$(H) \qquad 3^{\alpha_2} p_3^{\alpha_3} + 2^{\alpha_1} p_3^{\alpha_3} + 2^{\alpha_1} 3^{\alpha_2} + 2^{\alpha_1} 3^{\alpha_2} = 2^{\alpha_1} 3^{\alpha_2} p_3^{\alpha_3} - 1,$$

so that

$$(H') \qquad p_3^{\alpha_3} = \frac{2^{\alpha_1+1} 3^{\alpha_2} + 1}{2^{\alpha_1} 3^{\alpha_2} - 2^{\alpha_1} - 3^{\alpha_2}}.$$

If λ_0 is the largest of the four terms in the left member of (H)

$$4\lambda_0 > g \qquad \text{or} \qquad \lambda_0 > \frac{g}{4}.$$

That is, either one half or one third of the operations in g belong to one conjugate set. Hence either $\alpha_1 = 1$ or $\alpha_2 = 1$.

If $\alpha_1 = 1$, (H') becomes

$$p_3^{\alpha_3} = \frac{2^2 3^{\alpha_2} + 1}{3^{\alpha_2} - 2} = 4 + \frac{9}{3^{\alpha_2} - 2}.$$

Hence

$$\alpha_2 = 1, \ \alpha_3 = 1 \text{ and } g = 2 \cdot 3 \cdot 13.$$

It is easy to verify that no group G exists of this order.

If $\alpha_2 = 1$, (H') becomes

$$p_3^{\alpha_3} = \frac{2^{\alpha_1+1} 3 + 1}{2^{\alpha_1+1} - 3} = 3 + \frac{10}{2^{\alpha_1+1} - 3}.$$

From this, if $\alpha_1 > 1$, $p_3^{\alpha_3} = 5$ and $g = 2^2 \ 3 \ 5$

If, however, $\alpha_3 = 0$,

$$\frac{g}{2^{\alpha_1}} + \frac{g}{3^{\alpha_2}} + \frac{g}{3^{\beta_2}} = g - 1,$$

where

$$\beta_2 = \alpha_2 \text{ unless } \alpha_2 > 2.$$

But this equation cannot be satisfied when $\alpha_2 > 2$. Hence it may be written

$$3^{\alpha_2} + 2^{\alpha_1} + 2^{\alpha_1} = 2^{\alpha_1} 3^{\alpha_2} - 1,$$

$$3^{\alpha_2} = 2 + \frac{3}{2^{\alpha_1} - 1}.$$

Hence,

$$\alpha_1 = 2, \qquad \alpha_2 = 1 \qquad \text{and} \qquad g = 12.$$

Thus we have the result that $g = 3$, 10, 12, or 60. Of each of these orders there is just one such group.

The group of order 3, *the dihedron group of order* 10, *the tetrahedron group, and the icosahedron group are the only groups which have just one more complete set of conjugate operations, apart from identity, than there are distinct prime factors in the order of the group.*

THE UNIVERSITY OF ILLINOIS.

ON HYPERCOMPLEX NUMBER SYSTEMS*

(FIRST PAPER)

BY

HENRY TABER

Introduction.

The method invented by BENJAMIN PEIRCE† for treating the problem to determine all hypercomplex number systems (or algebras) in a given number of units depends chiefly, first, upon the classification of hypercomplex number systems into idempotent number systems, containing one or more idempotent numbers,‡ and non-idempotent number systems containing no idempotent number; and, second, upon the regularization of idempotent number systems, that is, the classification of each of the units of such a system with respect to one of the idempotent numbers of the system.

For the purpose of such classification and regularization the following theorems are required:

THEOREM I.§ *In any given hypercomplex number system there is an idempotent number (that is, a number $I \neq 0$ such that $I^2 = I$), or every number of the system is nilpotent.* ‖

THEOREM II.¶ *In any hypercomplex number system containing an idempotent number I, the units $e_1, e_2, \cdots, e_n$ can be so selected that, with reference*

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† American Journal of Mathematics, vol. 4 (1881), p. 97. This work, entitled *Linear Associative Algebra*, was published in lithograph in 1870. For an estimate of PEIRCE's work and for its relation to the work of STUDY, SCHEFFERS, and others, see articles by H. E. HAWKES in the American Journal of Mathematics, vol. 24 (1902), p. 87, and these Transactions, vol. 3 (1902), p. 312. The latter paper is referred to below when reference is made to HAWKES' work.

‡ If the hypercomplex number $A \neq 0$ and $A^2 = A$, PEIRCE terms A "idempotent," loc. cit., p. 104. See p. 512 below.

§ PEIRCE, loc. cit., p. 113; HAWKES, loc. cit., p. 321.

‖ If, for some positive integer m, $A^m = 0$, PEIRCE terms the hypercomplex number A "nilpotent," loc. cit., p. 104; see also p. 512 below.

¶ PEIRCE, loc. cit., p. 109; HAWKES, loc. cit., p. 314.

to I (termed the basis), they shall fall into one or other of four groups as follows:

$$1st\ group: \quad Ie_i = e_i = e_i I \qquad (i = 1, 2, \cdots, n'),$$

$$2d\ group: \quad Ie_i = e_i,\ e_i I = 0 \qquad (i = n'+1, n'+2, \cdots, n''),$$

$$3d\ group: \quad Ie_i = 0,\ e_i I = e_i \qquad (i = n''+1, n''+2, \cdots, n'''),$$

$$4th\ group: \quad Ie_i = 0 = e_i I \qquad (i = n'''+1, n'''+2, \cdots, n).$$

When so selected the units $e_1, e_2, \cdots, e_n$ of the number system are said to be *regular* with respect to I.* The units $e_1, e_2, \cdots, e_{n'}$ of the first group form a subsystem by themselves to which the idempotent number I belongs and of which I is the modulus; and I may, therefore, be chosen as one of the units of the first group.

THEOREM III.† *The basis or idempotent number I may be so chosen that there shall be no other idempotent number in the first group.*

THEOREM IV.‡ *In an idempotent number system § whose units are regular with respect to the basis or idempotent number I, chosen as one of the units, the remaining units of the first group can be taken to be nilpotent, ‖ and will then constitute a non-idempotent system by themselves, provided the first group contains no second idempotent number.*

Of these theorems requisite for the establishment of PEIRCE's method, his proofs of theorems I and IV are invalid. In a recent paper, cited above, by H. E. HAWKES, in these Transactions, he has fully established this chain of theorems, and thus, finally, placed PEIRCE's valuable method on a rigorous basis. But, whereas PEIRCE's methods are purely algebraic and involve only elementary considerations, HAWKES has recourse to the theory of transformation groups in establishing theorem IV, thus introducing conceptions unnecessary for the establishment of PEIRCE's theory and foreign to his methods. In the paper referred to, HAWKES has contributed a very valuable extension of PEIRCE's method in the conception of the regularization of all the units of a non-nilpotent number system with respect to each of certain idempotent numbers chosen as units.¶

The object of this paper at the outset was *to establish* PEIRCE's *method with-*

* HAWKES, loc. cit., p. 316.

† PEIRCE, loc. cit., p. 113; HAWKES, loc. cit., p. 316.

‡ PEIRCE, loc. cit., p. 117, 118; HAWKES, loc. cit., p. 322. This theorem may be expressed, otherwise, as follows. In a number system $e_1, e_2, \cdots, e_n$ with modulus e_n but no other idempotent number, the units $e_1, e_2, \cdots, e_{n-1}$ can be taken to be nilpotent and will then constitute by themselves a nilpotent system.

§ See p. 509; also p. 512.

‖ See note p. 509; also p. 512.

¶ HAWKES, loc. cit., p. 317; see also p. 512 below.

out recourse to the theory of groups. Employing methods purely algebraic, I give a proof of theorem I, which is, I think, somewhat simpler than that given by HAWKES; and I establish theorem IV by *the extension to number systems in general of the scalar function of quaternions;*—an extension suggested by my previous extension of the scalar function to matrices of any order,* and by C. S. PEIRCE's theorem that any number system can be represented by a matrix. †

The results of this paper in its original form were given at the February meeting of the American Mathematical Society of the current year; and the paper was to appear in the July number of these Transactions. But the employment of algebraic methods only, suggested to the editors an extension of the scope of the paper to the consideration of what may be designated as *number systems with respect to a domain R of rationality*, namely, the consideration of the totality of numbers

$$\sum_{i=1}^{n} a_i e_i$$

of a number system $e_1, e_2, \cdots, e_n$, where the a's are rational in the domain R of rationality of a given aggregate of real or ordinary complex numbers including the constants of multiplication of the system,—all transformations of the system being rational in R‡. At their request I have revised the paper from this point of view; and I give generalizations of all of PEIRCE's theorems with reference to a domain of rationality. § These results constitute a step towards *the enumeration of the types of number systems to which all number systems in a given number of units are reducible by transformations rational in a given domain.* I find that the conceptions on which PEIRCE's method is based are applicable to this problem. In a subsequent paper I shall give further results leading to a resolution of this problem.

*See Proceedings of the London Mathematical Society, vol. 22 (1891), p. 67, where I have applied this extension of the scalar function to a problem of SYLVESTER's.

†See Encyclopädie der Mathematischen Wissenschaften, vol. 1, p. 170, where this theorem is ascribed to ED. WEYR, though reference is made to C. S. PEIRCE's prior statement of the theorem in another form. It was established by C. S. PEIRCE in 1870 for certain of the number systems given in BENJAMIN PEIRCE's Linear Associative Algebra, and enunciated as probably true for number systems in general. See Memoirs of the American Academy of Arts and Sciences, new series, vol. 9 (1870), p. 363; also Johns Hopkins University Circulars, no. 22 (1883), p. 87. C. S. PEIRCE gave a demonstration of the general theorem in the Proceedings of the American Academy of Arts and Sciences, vol. 10 (1875), p. 392, and subsequently in the American Journal of Mathematics, vol. 4 (1881), p. 221.

In the development of the theory of the scalar function given below I have not employed C. S. PEIRCE's theorem, or the conception of a number system as represented by a matrix. See p. 514 below.

‡The arbitrary domain of rationality is not necessarily composed of real or ordinary complex numbers, e. g., the Galois fields are not. *The Editors.*

§In its original form, this paper treated a special case of such systems: namely, that for which the constants of multiplication of the system were real and R included every real scalar,—the case of real hypercomplex number systems.

The nomenclature employed by PEIRCE will be followed, in general, in this paper. Thus the first factor A in any product AB is termed the *facient*, the second factor B, the *faciend*:*

if $AB = A = BA$, A is an *idemfactor* with respect to B;
if $AB = A$, A is *idemfacient* with respect to B;
if $BA = A$, A is *idemfaciend* with respect to B;
if $AB = 0 = BA$, A is a *nilfactor* with respect to B;
if $AB = 0$, A is *nilfacient* with respect to B;
if $BA = 0$, A is *nilfaciend* with respect to B;
if $A \neq 0$, and $A^2 = A$, A is *idempotent*;
if $A^m = 0$ for some positive integer m, A is *nilpotent*;†

and a number system which contains no idempotent number is a *nilpotent system*.‡

A number system which contains one or more idempotent numbers is an *idempotent system*.§

The units of an idempotent system are *regular* with respect to the idempotent number I of the system when so chosen that each falls into one or other of PEIRCE's four groups, mentioned above, p. 510, with respect to I.§

A hypercomplex number A, satisfying the conditions fulfilled by the units of the kth group ($1 \leqq k \leqq 4$) with respect to the idempotent number I, is said to *belong* to the kth group with respect to I.

If A belongs to the k-th group with respect to I, it is linear in the units of that group, and conversely.‖ In particular, I belongs to the first group, and may be taken as one of the units of that group.

Following HAMILTON, the term *scalar* will be employed to denote a real or ordinary complex number.¶ If A is any given hypercomplex number, I shall designate by the term *polynomial in* A the hypercomplex number

$$c_1 A + c_2 A^2 + \cdots + c_p A^p$$

linear (with scalar coefficients) in positive integral powers of A. For any given number A of the system, there is at least one linear relation between the first $n+1$ power of A, and, therefore, a smallest integer $m \leqq n+1$ for which $A, A^2, \cdots, A^m$ are linearly related. This linear relation,

* These terms were suggested by HAMILTON, who employs the terms "factor" and "faciend" to denote A and B respectively, *Lectures on Quaternions*, p. 38.

† PEIRCE, loc. cit., p. 103–104.

‡ PEIRCE, loc. cit., p. 115.

§ HAWKES, loc. cit., pp. 314, 316.

‖ PEIRCE, loc. cit., p. 110. If $IA = A = AI$, A belongs to the first group with respect to I; if $IA = A$, $AI = 0$, A belongs to the second group with respect to I; etc. For the group of the non zero product of any two numbers or units belonging to the various groups, see PEIRCE, loc. cit., p. 111, and HAWKES, loc. cit., p. 316.

¶ *Lectures on Quaternions*, pp. 58, 664.

$$\Omega(A) \equiv A^m + p_1 A^{m-1} + \cdots + p_{m-2} A^2 + p_{m-1} A = 0,$$

I shall term the *fundamental equation of* A. Let

$$f(A) \equiv \sum_{q=1}^{p} c_q A^q \qquad (c_p \neq 0)$$

be any polynomial in A: *if* $f(A) = 0$, *then* $p \geqq m$ *and*

$$f(\lambda) \equiv \sum_{q=0}^{p-m} \gamma_q \lambda^q \Omega(\lambda)$$

for any scalar λ; *conversely, if*

$$f(\lambda) \equiv \sum_{q=0}^{p-m} c_q \lambda^q \Omega(\lambda),$$

then $f(A) = 0$.

The constants of multiplication of the units $e_1, e_2, \cdots, e_n$ of the number system will be denoted by γ_{ijk} for $i, j, k = 1, 2, \cdots n$, and thus

$$e_i e_j = \gamma_{ij1} e_1 + \gamma_{ij2} e_2 + \cdots + \gamma_{ijn} e_n;$$

and the domain of rationality of the constants γ_{ijk} will be denoted by $R(\gamma_{ijk})$. *The coefficients* $p_1, p_2, \cdots p_{m-1}$ *of the fundamental equation* $\Omega(A) = 0$ *of the hypercomplex number* $A = a_1 e_1 + \cdots + a_n e_n$ *are rational functions of* $a_1, a_2, \cdots a_n$ *for the domain* $R(\gamma_{ijk})$, being rational functions, for the domain $R(1)$, of $a_1, a_2, \cdots, a_n$ and of the constants γ_{ijk} of multiplication of the system.

Let R' denote the domain of rationality of any given aggregate of scalars. The totality of hypercomplex numbers

$$A = \sum_{i=1}^{n} a_i e_i,$$

for all possible sets of scalars $a_1, a_2, \cdots, a_n$ rational in R', will be said to constitute the *hypercomplex domain* $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$; and any such number A will be said to *belong to* $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$. *If* ρ *is any scalar rational in* R', *and both* A *and* B *belong to* $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, *so also does* ρA *and* $A + B$; *moreover,* AB *belongs to this domain provided* R' *contains* $R(\gamma_{ijk})$, but, in general, not otherwise. Therefore, if $c_1, c_2, \cdots, c_p$ are rational in R' and A belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, the polynomial

$$\sum_{q=1}^{p} c_q A^q$$

belongs to this domain if R' contains $R(\gamma_{ijk})$.

Let new units $e_1', e_2', \cdots, e_n'$ be introduced by the linear transformation T, and let γ_{ijk}' denote the constants of multiplication of the new system. Then,

if T is rational in R', the new units $e_1', e_2', \cdots, e_n'$ belong to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, and conversely;

*if T is rational in R', the hypercomplex domains $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$ and $\mathfrak{R}(R'; e_1', e_2', \cdots, e_n')$ are identical, and conversely;**

if T is rational in R', and R' contains $R(\gamma_{ijk})$, it also contains $R(\gamma'_{ijk})$.†

§ 1. *The scalar function of a hypercomplex number.*

Let

$$A = \sum_{i=1}^{n} a_i e_i$$

be any number of the given hypercomplex number system $e_1, e_2, \cdots, e_n$. I shall employ SA, in designation the *scalar* of A, to denote that function of the coefficients a and the constants γ_{ijk} of multiplication of the system‡ defined as follows:§

$$SA = \frac{1}{n}\left(a_1 \sum_{j=1}^{n} \gamma_{1jj} + a_2 \sum_{j=1}^{n} \gamma_{2jj} + \cdots + a_n \sum_{j=1}^{n} \gamma_{njj} \right). \tag{1}$$

Whence, if ρ is any scalar,‖ and

$$B = \sum_{i=1}^{n} b_i e_i$$

is any second number of the system, it follows that

$$S\rho A = \rho SA, \tag{2}$$

---

* In order that T shall be rational in R' it is only necessary that $\mathfrak{R}(R'; e_1, e_2, e_n)$ shall contain $\mathfrak{R}(R'; e_1', e_2', \cdots, e_n')$.

† In particular R' may be identical with $R(\gamma_{ijk})$, in which case, if T is rational in $R(\gamma_{ijk})$, this domain contains $R(\gamma'_{ijk})$; but, unless the coefficients of T are rational in $R(1)$, $R(\gamma'_{ijk})$ is, in general, not identical with $R(\gamma_{ijk})$. Thus, if

$$e_1 = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

e_1, e_2 constitute a number system, and $R(\gamma_{ijk})$ contains $\sqrt{2}$; and, thus, the transformation

$$e_1' = \frac{1}{\sqrt{2}} \cdot e_1, \qquad e_2' = e_2$$

is rational in $R(\gamma_{ijk})$. But the constants of multiplication of the new system are rational in $R(1)$; and, therefore, $R(\gamma'_{ijk})$ is not identical with $R(\gamma_{ijk})$.

‡ See p. 513 above.

§ If $n=4$, and e_1, e_2, e_3, e_4 are the quaternion units, this definition accords with the customary definition of the scalar function. Thus, if $e_4 = 1$ and e_1, e_2, e_3 constitute a system of mutually normal vectors, we have

$$\gamma_{4jj} = 1, \qquad \gamma_{ijj} = 0 \qquad (i = 1, 2, 3; j = 1, 2, 3, 4)$$

and thus

$$S\sum_{i=1}^{4} a_i e_i = a_4$$

by the above definition.

‖ See p. 512 above.

(3) $$S(A+B) = SA + SB.$$

In particular,

(4) $$SA = a_1 Se_1 + a_2 Se_2 + \cdots + a_n Se_n.$$

Moreover, if

$$\epsilon = \sum_{i=1}^{n} a_i^{(0)} e_i$$

is a modulus of the system, in which case

$$\sum_{i=1}^{n} a_i^{(0)} e_i e_j = \epsilon e_j = e_j \qquad (j=1, 2, \cdots, n),$$

we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i^{(0)} \gamma_{ijj} = n,$$

and therefore

(5) $$S\epsilon = 1.$$

Let R' denote the domain of rationality of any given aggregation of scalars that includes the constants γ_{ijk} of multiplication of the system. If A belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$,* SA is rational in R'. In particular, let R' include every real scalar, and let $e_1, e_2, \cdots, e_n$ be the units of a real hypercomplex number system† in which case the constants γ_{ijk} of multiplication of the system are real: then, if A is a number of the system, in which case $a_1, a_2, \cdots, a_n$ are real, SA is real.

By definition

(6) $$SAB = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{h=1}^{n} a_i b_j \gamma_{ijk} \gamma_{khh},$$

$$SBA = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{h=1}^{n} a_j b_i \gamma_{ijk} \gamma_{khh}.$$

But, since

$$e_i e_j \cdot e_h = e_i \cdot e_j e_h \qquad (i, j, h=1, 2, \cdots, n),$$

we have

(6′) $$\sum_{k=1}^{n} \gamma_{ijk} \gamma_{khl} = \sum_{k=1}^{n} \gamma_{ikl} \gamma_{jhk} \qquad (i, j, h, l=1, 2, \cdots, n);$$

therefore,

(7) $$\begin{aligned} SAB &= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{h=1}^{n} a_i b_j \gamma_{ijk} \gamma_{khh} \\ &= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{h=1}^{n} a_i b_j \gamma_{ikh} \gamma_{jhk} \\ &= \frac{1}{n} \sum_{i=1}^{n} \sum_{=1}^{n} \sum_{k=1}^{n} \sum_{h=1}^{n} a_j b_i \gamma_{ikh} \gamma_{jhk} \ddagger \end{aligned}$$

* See p. 513 above.

† See note p. 511 above.

‡ By interchange of i and j, and of k and h.

$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{h=1}^{n} a_j b_i \gamma_{ijk} \gamma_{khh} = SBA .$$

Let now $e_1', e_2', \cdots, e_n'$ be any second system of units of the number system obtained from $e_1, e_2, \cdots, e_n$ by the transformation T of non-zero determinant defined as follows:

(8) $$e_i' = \tau_{i1} e_1 + \tau_{i2} e_2 + \cdots + \tau_{in} e_n \qquad (i = 1, 2, \cdots, n);$$

and let

(9) $$e_i' e_j' = \gamma_{ij1}' e_1' + \gamma_{ij2}' e_2 + \cdots + \gamma_{ijn}' e_n \qquad (i, j = 1, 2, \cdots, n).$$

Let

$$a_1' e_1' + a_2' e_2' + \cdots + a_n' e_n' = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n,$$

$$b_1' e_1' + b_2' e_2' + \cdots + b_n' e_n' = b_1 e_2 + b_2 e_2 + \cdots + b_n e_n;$$

then

$$\left.\begin{aligned} a_k &= \tau_{1k} a_1' + \tau_{2k} a_2' + \cdots + \tau_{nk} a_n', \\ b_k &= \tau_{1k} b_1' + \tau_{2k} b_2' + \cdots + \tau_{nk} b_n', \end{aligned}\right\} \qquad (k = 1, 2, \cdots, n),$$

that is, in CAYLEY's abbreviated notation *

$$(a_1, a_2, \cdots a_n) = (T \{ a_1', a_2', \cdots a_n'),$$

$$(b_1, b_2, \cdots b) = (T \{ b_1', b_2', \cdots b_n'),$$

where T is the matrix

$$\begin{Vmatrix} \tau_{11} & \tau_{21} & \cdots & \tau_{n1} \\ \tau_{12} & \tau_{22} & \cdots & \tau_{n2} \\ \cdot & \cdot & \cdots & \cdot \\ \tau_{1n} & \tau_{2n} & \cdots & \tau_{nn} \end{Vmatrix}.$$

Finally, let

$$\beta_1' e_1' + \beta_2' e_2' + \cdots + \beta_n' e_n' = (a_1' e_1' + a_2' e_2' + \cdots + a_n' e_n')(b_1' e_1' + b_2' e_2' + \cdots + b_n' e_n'),$$

$$\beta_1 e_1 + \beta_2 e_2 + \cdots + \beta_n e_n = (a_1 e_1 + a_2 e_2 + \cdots + a_n e_n)(b_1 e_1 + b_2 e_2 + \cdots + b_n e_n).$$

Then

$$(\beta_1', \beta_2', \cdots, \beta_n') = \begin{Vmatrix} \sum_{i=1}^{n} a_i' \gamma_{i11}' & \sum_{i=1}^{n} a_i' \gamma_{i21}' & \cdots & \sum_{i=1}^{n} a_i' \gamma_{in1}' \\ \sum_{i=1}^{n} a_i' \gamma_{i12}' & \sum_{i=1}^{n} a_i' \gamma_{i22} & \cdots & \sum_{i=1}^{n} a_i' \gamma_{in2} \\ \cdot & \cdot & \cdots & \cdot \\ \sum_{i=1}^{n} a_i' \gamma_{i1n}' & \sum_{i=1}^{n} a_i' \gamma_{i2n} & \cdots & \sum_{i=1}^{n} a_i' \gamma_{inn} \end{Vmatrix} (b_1', b_2', \cdots, b_n'),$$

* *Memoir on Matrices*, Philosophical Transactions, vol. 148 (1858), p. 17.

$$(\beta_1, \beta_2, \cdots, \beta_n) = \begin{vmatrix} \sum_{i=1}^{n} a_i\gamma_{i11} & \sum_{i=1}^{n} a_i\gamma_{i21} & \cdots & \sum_{i=1}^{n} a_i\gamma_{in1} \\ \sum_{i=1}^{n} a_i\gamma_{i12} & \sum_{i=1}^{n} a_i\gamma_{i22} & \cdots & \sum_{i=1}^{n} a_i\gamma_{in2} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^{n} a_i\gamma_{i1n} & \sum_{i=1}^{n} a_i\gamma_{i2n} & \cdots & \sum_{i=1}^{n} a_{inn} \end{vmatrix} (b_1, b_2, \cdots, b_n);$$

moreover,

$$\beta'_1 e'_1 + \beta'_2 e'_2 + \cdots + \beta'_n e'_n = \beta_1 e_1 + \beta_2 e_2 + \cdots + \beta_n e_n,$$

and, therefore,

$$(\beta_1, \beta_2, \cdots, \beta_n) = (T)(\beta'_1, \beta'_2, \cdots, \beta'_n).$$

Wherefore,

$$\begin{vmatrix} \sum_{i=1}^{n} a'_i\gamma'_{i11} & \sum_{i=1}^{n} a'_i\gamma'_{i21} & \cdots & \sum_{i=1}^{n} a'_i\gamma'_{in1} \\ \sum_{i=1}^{n} a'_i\gamma'_{i12} & \sum_{i=1}^{n} a'_i\gamma'_{i22} & \cdots & \sum_{i=1}^{n} a'_i\gamma'_{in2} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^{n} a'_i\gamma'_{i1n} & \sum_{i=1}^{n} a'_i\gamma'_{i2n} & \cdots & \sum_{i=1}^{n} a'_i\gamma'_{inn} \end{vmatrix}$$

(10)

$$= T^{-1} \begin{vmatrix} \sum_{i=1}^{n} a_i\gamma_{i11} & \sum_{i=1}^{n} a_i\gamma_{i21} & \cdots & \sum_{i=1}^{n} a_i\gamma_{n1} \\ \sum_{i=1}^{n} a_i\gamma_{i12} & \sum_{i=1}^{n} a_i\gamma_{i22} & \cdots & \sum_{i=1}^{n} a_i\gamma_{in2} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^{n} a_i\gamma_{i1n} & \sum_{i=1}^{n} a_i\gamma_{i2n} & \cdots & \sum_{i=1}^{n} a_i\gamma_{inn} \end{vmatrix} T.$$

Whence it follows that the characteristic equations of these two matrices are identical, that is,

$$\begin{vmatrix} \lambda - \sum_{i=1}^{n} a'_i\gamma'_{i11} & -\sum_{i=1}^{n} a'_i\gamma'_{i21} & \cdots & -\sum_{i=1}^{n} a'_i\gamma'_{in1} \\ -\sum_{i=1}^{n} a'_i\gamma'_{i12} & \lambda - \sum_{i=1}^{n} a'_i\gamma'_{i22} & \cdots & -\sum_{i=1}^{n} a'_i\gamma'_{in2} \\ \cdots & \cdots & \cdots & \cdots \\ -\sum_{i=1}^{n} a'_i\gamma'_{i1n} & -\sum_{i=1}^{n} a'_i\gamma'_{i2n} & \cdots & \lambda - \sum_{i=1}^{n} a'_i\gamma'_{inn} \end{vmatrix}$$

(11)

$$(11) \qquad \equiv \begin{vmatrix} \lambda - \sum_{i=1}^{n} a_i\gamma_{i11} & -\sum_{i=1}^{n} a_i\gamma_{i21} & \cdots & -\sum_{i=1}^{n} a_i\gamma_{in1} \\ -\sum_{i=1}^{n} a_i\gamma_{i12} & \lambda - \sum_{i=1}^{n} a_i\gamma_{i22} & \cdots & -\sum_{i=1}^{n} a_i\gamma_{in2} \\ \cdots & \cdots & \cdots & \cdots \\ -\sum_{i=1}^{n} a_i\gamma_{i1n} & -\sum_{i=1}^{n} a_i\gamma_{i2n} & \cdots & \lambda - \sum_{i=1}^{n} a_i\gamma_{inn} \end{vmatrix}.$$

Therefore, in particular,

$$(12) \qquad \sum_{i=1}^{n}\sum_{j=1}^{n} a_i'\gamma_{ijj}' = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i\gamma_{ijj};$$

and, thus, SA is an invariant to any transformation of the units of the number system.

Let

$$I = \sum_{i=1}^{n} a_i^{(0)} e_i$$

be any idempotent number* of the system $e_1, e_2, \cdots, e_n$; and, let

$$(13) \qquad e_i^{(1)} = Ie_i, \qquad e_i^{(2)} = e_i - Ie_i \qquad (i = 1, 2, \cdots, n).$$

We have

$$(14) \qquad \begin{aligned} Ie_i^{(1)} &= I^2 e_i = Ie_i = e_i^{(1)}, \\ Ie_i^{(2)} &= Ie_i - I^2 e_i = Ie_i - Ie_i = 0 \end{aligned} \qquad (i = 1, 2, \cdots, n).$$

Thus, for $1 \leqq i \leqq n$, the numbers $e_i^{(1)}$, $e_i^{(2)}$ are, respectively, idemfaciend and nilfaciend† with respect to I. Further, any number linear in the numbers $e_i^{(1)}$ is idemfaciend with respect to I, and any number linear in the numbers $e_i^{(2)}$ is nilfaciend with respect to I. Since

$$(15) \qquad e_i = e_i^{(1)} + e_i^{(2)} \qquad (i = 1, 2, \cdots, n),$$

every number of the system can be expressed linearly in the $2n$ numbers of (13). Let n_1 of the numbers $e_i^{(1)}$, as $e_1^{(1)}, e_2^{(1)}, \cdots, e_{n_1}^{(1)}$ be independent; and let n_2 of the numbers $e_i^{(2)}$, as $e_1^{(2)}, e^{(2)}, \cdots, e_{n_2}^{(2)}$ be independent. Then since, for $k = 1, 2$ and $i = 1, 2, \cdots, n$, each of the numbers $e_i^{(k)}$ (and, therefore, any n independent numbers of the system) can be expressed in terms of the above $n_1 + n_2$ hypercomplex numbers, and the latter are independent,‡ we have $n_1 + n_2 = n$. We may now

* See p. 512 above.

† See p. 512 above.

‡ For, if $c_i^{(1)}e_1^{(1)} + \cdots + c_{n_1}^{(1)}e_{n_1}^{(1)} + c_1^{(2)}e_1^{(2)} + \cdots + c_{n_2}^{(2)}e_{n_2}^{(2)} = 0$, then

$$c_1^{(1)}e_1^{(1)} + \cdots + c_{n_1}^{(1)}e_{n_1}^{(1)} = I(c_1^{(1)}e_1^{(1)} + \cdots + c_{n_1}^{(1)}e_{n_1}^{(1)} + c_1^{(2)}e_1^{(2)} + \cdots + c_{n_2}^{(2)}e_{n_2}^{(2)}) = 0;$$

and, therefore, $c_1^{(1)} = \cdots = c_{n_1}^{(1)} = c_1^{(2)} = \cdots = c_{n_2}^{(2)} = 0$.

show that any number idemfaciend with respect to I is linear in $e_1^{(1)}, e_2^{(1)}, \cdots, e_{n_1}^{(1)}$; whence it follows that I is expressible in these hypercomplex numbers. Further, that any number nilfaciend to I is linear in $e_1^{(2)}, e_2^{(2)}, \cdots, e_{n_2}^{(2)}$. It may be that $n_2 = 0$; but $n_1 \geqq 1$. For n_1 is the number of independent hypercomplex numbers in the aggregate Ie_1 for $i = 1, 2, \cdots, n$; and the n numbers of this aggregate cannot all be zero, that is, $n_1 \neq 0$, otherwise,

$$I = I^2 = I\sum_{i=1}^{n} a_i^{(0)} e_i = \sum_{i=1}^{n} a_i^{(0)} Ie_i = 0.$$

Of the numbers $e_i^{(1)}$, $n_1 - 1$ are independent of I. Let $e_1^{(1)}, e_2^{(1)}, \cdots, e_{n_1-1}^{(1)}$ be independent of I. Then the n hypercomplex numbers $I, e_i^{(1)}$ $(i=1, 2, \cdots\ n_1-1)$, and $e_i^{(2)}$ $(i = 1, 2, \cdots, n_2)$ are independent. Let

$$\begin{aligned} e_1' &= I, \\ e_{1+i}' &= e_i^{(1)} && (i = 1, 2, \cdots, n_1-1), \\ e_{n_1+i}' &= e_i^{(2)} && (i = 1, 2, \cdots, n_2); \end{aligned} \tag{16}$$

and let γ_{ijk}' denote the constants of multiplication of this new system of units, so that

$$e_i' e_j' = \sum_{k=1}^{n} \gamma_{ijk}' e_k'.$$

Since

$$\begin{aligned} e_1' e_j' &= e_j' && (j = 1, 2, \cdots, n_1), \\ e_1' e_j' &= 0 && (j = n_1+1, n_1+2, \cdots, n), \end{aligned} \tag{17}$$

we have

$$\begin{aligned} \gamma_{1jj}' &= 1 && (j = 1, 2, \cdots, n_1), \\ \gamma_{1jj}' &= 0 && (j = n_1+1, n_1+2, \cdots, n); \end{aligned} \tag{18}$$

and, therefore, since SI is invariant to any transformation of the units of the system,

$$SI = Se_1' = \frac{1}{n}\sum_{i=1}^{n} \gamma_{1ij}' = \frac{n_1}{n}. \tag{19}$$

If I belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, then, since each of the original system of units belongs to this domain, it follows that each of the hyper-complex numbers $e_i^{(k)}$, for $k = 1, 2$ and $i = 1, 2, \cdots, n_k$, belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$.* Therefore, if I belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, $n_1 = nSI$ is the number of independent hypercomplex numbers belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$ which are idemfaciend to I; and $n_2 = n(1 - SI)$ is the number of independent hypercomplex numbers belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$ which are nilfaciend to I.

Let N be any nilpotent† number, other than zero, of the system $e_1, e_2, \cdots, e_n$.

* See p. 513 above.

† See p. 512 above.

Thus, let $N^{m-1} \neq 0$ and $N^m = 0$. For any positive integer $p < m$, there is at least one number of the system, as N^{m-p}, to which N^p, but no lower power of N, is nilfacient;* and any such number is independent of the numbers to which N^q is nilfacient for $q < p$, since, if N^q is nilfacient to A_1, A_2, etc., it is nilfacient to any number linear in A_1, A_2, etc. If for $1 < p \leqq m$, N^p, but no lower power of N, is nilfacient to A_1', A_2', etc., and if these hypercomplex numbers are independent of each other and of all hypercomplex numbers to which N^q is nilfacient for $q < p$, we then cannot have

$$N^{p'}(c_1 A_1' + c_2 A_2' + \cdots) = 0,$$

for $p' < p$, unless the c's are all zero; otherwise, there is a linear relation between the A''s and the hypercomplex numbers to which $N^{p'}$ is nilfacient. Let the system contain just $n_1 \geqq 1$ independent numbers, as

$$A_1^{(1)}, A_2^{(1)}, \cdots, A_{n_1}^{(1)},$$

to which N is nilfacient. Then, if $NB = 0$, we have

$$B = \sum_{j=1}^{n_1} c_j A_j^{(1)}.$$

If $m > 2$, let the system contain just $n_2 \geqq 1$ numbers, as

$$A_1^{(2)}, A_2^{(2)}, \cdots, A_{n_2}^{(2)},$$

to which N^2 is nilfacient, but N is not nilfacient, and which are independent, moreover, of each other and the $A^{(1)}$'s† (and, therefore, of numbers to which N is nilfacient). Then, if $NB \neq 0$, and $N^2B = 0$, we have

$$B = \sum_{j=1}^{n_1} c_j^{(1)} A_j^{(1)} + \sum_{j=1}^{n_2} c_j^{(2)} A_j^{(2)},$$

where the c's with index 2 are not all zero. If $m > 3$, let the system contain just $n_3 \geqq 1$ numbers, as

$$A_1^{(3)}, A_2^{(3)}, \cdots, A_{n_3}^{(3)},$$

to which N^3, but no lower power of N, is nilfacient, and which are independent moreover, of each other and of the $A^{(1)}$'s and $A^{(2)}$'s (and, therefore, of numbers to which N^2 is nilfacient). Then, as before, if $N^2B \neq 0$, $N^3B = 0$, we have

$$B = \sum_{j=1}^{n_1} c_j^{(1)} A_j^{(1)} + \sum_{j=1}^{n_2} c_j^{(2)} A_j^{(2)} + \sum_{j=1}^{n_3} c_j^{(3)} A_j^{(3)},$$

*See p. 512 above.

† By this I mean, not only that the $A^{(2)}$'s are independent of each other, and, severally of the $A^{(1)}$'s, but that, if

$$\sum_{j=1}^{n_1} c_j^{(1)} A_j^{(1)} + \sum_{j=1}^{n_2} c_j^{(2)} A_j^{(2)} = 0;$$

the coefficients c are then all zero.

where the c's with index 3 are not all zero. Etc., etc. Finally, let the system contain just n_{m-1} numbers, as

$$A_1^{(m-1)}, A_2^{(m-1)}, \cdots, A_{n_{m-1}}^{(m-1)},$$

to which N^{m-1}, but no lower power of N, is nilfacient, and which are independent, moreover, of each other and of the $A^{(1)}$'s, $A^{(2)}$'s, $\cdots$, $A^{(m-2)}$'s (and, therefore, of numbers to which N^{m-2} is nilfacient). Then, if $N^{m-2}B \neq 0$, $N^{m-1}B = 0$, we have

$$B = \sum_{j=1}^{n_1} c_j^{(1)} A_j^{(1)} + \sum_{j=1}^{n_2} c_j^{(2)} A_j^{(2)} + \cdots + \sum_{j=1}^{n_{m-1}} c_j^{(m-1)} A_j^{(n_{m-1})},$$

where the c's with index $m-1$ are not all zero. We may take N as one of the $A^{(m-1)}$: thus let $A_{n_{m-1}}^{(m-1)} = N$. Let now the system contain just $n_m \geqq 0$ numbers, as

$$A_1^{(m)}, A_2^{(m)}, \cdots, A_{n_m}^{(m)},$$

independant of each other and of the $A^{(1)}$'s, $A^{(2)}$'s etc., and $A^{(m-1)}$'s (and, therefore, of numbers to which N^{m-1} is nilfacient). Then if $N^{m-1}B \neq 0$, we have

$$B = \sum_{j=1}^{n_1} c_j^{(1)} A_j^{(1)} + \sum_{j=1}^{n_2} c_j^{(2)} A_j^{(2)} + \cdots + \sum_{j=1}^{n_m} c_j^{(m)} A_j^{(m)},$$

where the c's with index m are not all zero. By definition,

$$NA_i^{(1)} = 0 \qquad (i = 1, 2, \cdots, n_1); \tag{20}$$

and, by what precedes, for $1 < p \leqq m$,

$$NA_i^{(p)} = \sum_{j=1}^{n_1} \gamma_{ij}^{(1)} A_j^{(1)} + \sum_{j=1}^{n_2} \gamma_{ij}^{(2)} A_j^{(2)} + \cdots + \sum_{j=1}^{n_{p-1}} \gamma_{ij}^{(p-1)} A_j^{(p-1)} \quad (i = 1, 2, \cdots, n_p), \tag{21}$$

since N^{p-1} is nilfacient to $NA_i^{(p)}$.* Moreover, every number of the system $e_1, e_2, \cdots, e_n$ is linear in the $A^{(1)}$'s, $A^{(2)}$'s, $\cdots$, $A^{(m)}$'s, since with respect to

* Let $1 < p \leqq m$. Since, by supposition, the $A^{(p)}$'s are independent of each other and of all numbers to which N^{p-1} is nilfacient we cannot have

$$N^{p-1}(c_1 A_1^{(p)} + c_2 A_2^{(p)} + \cdots + c_{n_p} A_{n_p}^{(p)}) = 0,$$

unless the c's are all zero. Therefore,

$$N^{p-2}(c_1 NA_1^{(p)} + c_2 NA_2^{(p)} + \cdots + c_{n_p} NA_{n_p}^{(p)}) \neq 0,$$

unless the c's are all zero, that is, the numbers $NA_i^{(p)}$ $(i = 1, 2, \cdots, n_p)$ are independent of each other and of all numbers to which N^{p-2} is nilfacient; moreover,

$$N^{p-1} \cdot NA_i^{(p)} = N^p A_i^{(p)} = 0 \qquad (i = 1, 2, \ldots, n_p)$$

But, by supposition, there are but n_{p-1} numbers to which N^{p-1} is nilfacient, and which are independent of each other and of all numbers to which N^{p-2} is nilfacient. Therefore, $n_p \leqq n_{p-1}$. Whence it follows that we may put $NA_i^{(p)} = A_i^{(p-1)}$ $(i = 1, 2, \cdots, n_p)$.

every number B of the system either $N^{m-1}B \neq 0$, or $N^pB = 0$ for $1 \leqq p < m$. Therefore, new units $e_1', e_2', \cdots, e_n'$ may be introduced by the transformation,

$$
\begin{aligned}
e_i' &= A_i^{(1)} && (i = 1, 2, \cdots, n_1), \\
e_{n_1+i}' &= A_i^{(2)} && (i = 1, 2, \cdots, n_2), \\
&\cdots\cdots\cdots && \\
e_{n_1+n_2+\cdots+n_{m-1}+i}' &= A_i^{(m)} && (i = 1, 2, \cdots, n_m).
\end{aligned} \tag{22}
$$

Since

$$
N = A_{n_{m-1}}^{(m-1)} = e_{n_1+n_2+\cdots+n_{m-1}}', \tag{23}
$$

if γ_{ijk}' denote the constants of multiplication of the new units, we have, by (20) and (21),

$$
\gamma_{n_1+n_2+\cdots n_{m-1},jj}' = 0 \qquad (j = 1, 2, \cdots, n). \tag{24}
$$

Therefore, since SA is invariant to any transformation of the units of the system,

$$
SN = SA_{n_{m-1}}^{(m-1)} = Se_{n_1+n_2+\cdots+n_{m-1}}' = \sum_{j=1}^{n} \gamma_{n_1+n_2+\cdots+n_{m-1},jj}' = 0. \tag{25}
$$

That is, the scalar function of any nilpotent number A of the system is equal to zero. But if A is nilpotent, so also are N^2, N^3, etc. Therefore, if N is nilpotent, $SN^p = 0$ for any positive integer p.

Let the number system contain n independent numbers, $A_1, A_2, \cdots, A_n$, such that

$$
SA_1 = SA_2 = \cdots = SA_n = 0.
$$

For any number A of the system, we have

$$
A = c_1 A_1 + c_2 A_2 + \cdots + c_n A_n,
$$

and, therefore, by (2) and (3),

$$
SA = c_1 SA_1 + c_2 SA_2 + \cdots + c_n SA_n = 0.
$$

In this case the system is nilpotent;* that is, contains no idempotent number, since, if I is idempotent, $SI \neq 0$. In particular, if

$$
Se_1 = Se_2 = \cdots = Se_n = 0,
$$

that is, if

$$
\sum_{j=1}^{n} \gamma_{ijj} = 0 \qquad (i = 1, 2, \cdots, n),
$$

the system is nilpotent.

These results may be summarized in the following theorem:

THEOREM (1). *Let*

$$
A = \sum_{i=1}^{n} a_i e_i
$$

* See p. 512 above.

be any number of the hypercomplex number system $e_1, e_2, \cdots, e_n$; *let* R' *denote any scalar domain of rationality including* $R(\gamma_{ijk})$;* *and let*

$$SA \equiv \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \gamma_{ijj}.$$

Then SA *is invariant to any transformation of the units of the system; and, if* ρ *is any scalar and* B *is any second number of the system,*

$$S\rho A = \rho SA,$$

$$S(A + B) = SA + SB,$$

$$SAB = SBA.$$

If ϵ *is a modulus of the system,* $S\epsilon = 1$. *If* A *belongs to* $\mathfrak{R}(R'; e_1, e_2, \cdots e_n)$, SA *is rational in* R'; *and if, moreover,* A *is idempotent,* $nSA \neq 0$ *is the number of independent hypercomplex numbers of the system, belonging to* $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, *that are idemfaciend to* A, *and* $n(1 - SA) \equiv 0$ *is the number of independent hypercomplex numbers of the system, belonging to* $\mathfrak{R}(R'; e_1, e_2, \cdots)$, *that are nilfaciend to* A. *If the system contains any* n *independent numbers* $A_1, A_2, \cdots, A_n$ *such that*

$$SA_1 = SA_2 = \cdots SA_n = 0,$$

in particular, if

$$Se_1 = Se_2 = \cdots = Se_n = 0,$$

the system is nilpotent. Finally, if A *is nilpotent,* $SA^p = 0$ *for any positive integer* p; *and, therefore, if the system contains* n *independent nilpotent numbers* $A_1, A_2, \cdots, A_n$, *in particular, if each of the units* $e_1, e_2, \cdots, e_n$ *is nilpotent, the system is nilpotent.*

Let the fundamental equation of A† be

$$\Omega(A) \equiv A^m + p_1 A^{m-1} + \cdots + p_{m-1} A = 0.$$

By the aid of this equation we can express A^p for $p \equiv m$ linearly in $A, A^2, \cdots, A^{m-1}$, and consequently, any polynomial in A‡ linearly in the first $m - 1$ powers of A. Therefore, by (2) and (3), *if the fundamental equation of* A *is of order* m *and*

$$SA = SA^2 = \cdots = SA^{m-1} = 0,$$

the scalar of any polynomial

$$\sum_{q=1}^{p} c_q A^q$$

in A *is equal to zero.*

*See p. 513 above.
†See p. 513 above.
‡See p. 512 above.

§ 2. *The idempotent number* $\mathfrak{f}(A)$.

Let A be any number of the hypercomplex number system $e_1, e_2, \cdots, e_n$; and let

$$\Omega(A) \equiv A^m + p_1 A^{m-1} + \cdots + p_{m-2} A^2 + p_{m-1} A = 0 \tag{26}$$

be the fundamental equation of A. Either

$$p_1 = p_2 = \cdots = p_{m-1} = 0$$

and $A^m = 0$, or the roots of the equation

$$\Omega(\lambda) \equiv \lambda^m + p_1 \lambda^{m-1} + \cdots + p_{m-2} \lambda^2 + p_{m-1} \lambda = 0 \tag{27}$$

are not all zero. Let $A^m \neq 0$; and let zero be a root of multiplicity $\nu < m$ of equation (27). Let the distinct roots, other than zero, of this equation be $\lambda_1, \lambda_2, \cdots, \lambda_r$, respectively, of multiplicity $\nu_1, \nu_2, \cdots, \nu_r$; and let $p_1^{(\nu)}, p_2^{(\nu)}, \cdots, p_{m-\nu}^{(\nu)}$, with alternate negative and positive sign, denote the simple symmetric functions of the νth powers of the non-zero roots of $\Omega(\lambda) = 0$, $\lambda_1^\nu, \lambda_2^\nu, \cdots, \lambda_r^\nu$ being counted, respectively, $\nu_1, \nu_2, \cdots, \nu_r$ times. Then, since

$$(\lambda^\nu)^{m-\nu+1} + p_1^{(\nu)}(\lambda^\nu)^{m-\nu} + \cdots + p_{m-\nu-1}^{(\nu)}(\lambda^\nu)^2 + p_{m-\nu}^{(\nu)}(\lambda^\nu) \equiv \lambda^\nu(\lambda^\nu - \lambda_1^\nu)^{\nu_1}(\lambda^\nu - \lambda_2^\nu)^{\nu_2} \cdots (\lambda^\nu - \lambda_r^\nu)^{\nu_r}$$

contains $\Omega(\lambda)$ as a factor, it follows that

$$(A^\nu)^{m-\nu+1} + p_1^{(\nu)}(A^\nu)^{m-\nu} + \cdots + p_{m-\nu-1}^{(\nu)}(A^\nu)^2 + p_{m-\nu}^{(\nu)}(A^\nu) = 0,^*$$

where

$$p_{m-\nu}^{(\nu)} = \pm(\lambda_1^{\nu_1}\lambda_2^{\nu_2} \cdots \lambda_r^{\nu_r})^\nu \neq 0.$$

Let now

$$\mathfrak{f}(A) \equiv \frac{1}{-p_{m-\nu}^{(\nu)}}[(A^\nu)^{m-\nu} + p_1^{(\nu)}(A^\nu)^{m-\nu-1} + \cdots + p_{m-\nu-1}^{(\nu)}(A^\nu)]. \tag{28}$$

We then have

$$A^\nu \mathfrak{f}(A) - A^\nu = \frac{1}{-p_{m-\nu}^{(\nu)}}[(A^\nu)^{m-\nu+1} + p_1^{(\nu)}(A^\nu)^{m-\nu} + \cdots + p_{m-\nu-1}^{(\nu)}(A^\nu)^2 + p_{m-\nu}^{(\nu)}(A^\nu)] = 0.$$

Therefore,

$$A^\nu \mathfrak{f}(A) = A^\nu = \mathfrak{f}(A) A^\nu; \tag{29}$$

and, thus, for any positive integer p,

$$A^{\nu+p} \mathfrak{f}(A) = A^{\nu+p} = \mathfrak{f}(A) A^{\nu+p}. \tag{30}$$

In particular, for any positive integer q,

$$(A^\nu)^q \mathfrak{f}(A) = (A^\nu)^q = \mathfrak{f}(A)(A^\nu)^q; \tag{31}$$

and, therefore,

$$\mathfrak{f}(A)\mathfrak{f}(A) = \mathfrak{f}(A). \tag{32}$$

---

* See p. 513 above.

The polynomial $\mathfrak{f}(A) \neq 0$; for, otherwise, by (29) $A^\nu = 0$, and, therefore, $A^m = 0$, which is contrary to supposition. By (31), the hypercomplex number $\mathfrak{f}(A)$ is a modulus of the number system whose units are the independent powers of A^ν. We have

$$(33) \qquad \mathfrak{f}(\lambda) \equiv 1 - \left(\frac{\lambda^\nu - \lambda_1^\nu}{-\lambda_1^\nu}\right)^{\nu_1}\left(\frac{\lambda^\nu - \lambda_2^\nu}{-\lambda_2^\nu}\right)^{\nu_2}\cdots\left(\frac{\lambda^\nu - \lambda_r^\nu}{-\lambda_r^\nu}\right)^{\nu_r};$$

in particular, if $\nu = 1$,

$$(33a) \qquad \mathfrak{f}(\lambda) \equiv \frac{1}{-p_{m-1}}(\lambda^{m-1} + p_1\lambda^{m-2} + \cdots + p_{m-2}\lambda).$$

Since the coefficients of the polynomial $\mathfrak{f}(A)$ in A are rational functions of $p_1, p_2, \cdots, p_{m-1}$ for the domain $R(1)$, they are rational functions in the same domain of $a_1, a_2, \cdots, a_n$ and of the constants γ_{ijk} of multiplication of the number system;* and, thus, are rational functions of $a_1, a_2, \cdots, a_n$ in the domain $R(\gamma_{ijk})$. Wherefore, if A belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, so also does $\mathfrak{f}(A)$, provided R' includes $R(\gamma_{ijk})$.† In particular, if $e_1, e_2, \cdots, e_n$ are the units of a real hypercomplex number system, and A is a number of this system, so also is $\mathfrak{f}(A)$.

Let R' include $R(\gamma_{ijk})$. Each of the units $e_1, e_2, \cdots e_n$ belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots e_n)$. Therefore, either one, at least, of the units, as e_k, is not nilpotent, in which case, if $A = e_k$ there is an idempotent number $\mathfrak{f}(A)$ belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$,—or each of the units is nilpotent, in which case by theorem (1) the system is nilpotent.

We have, therefore, the following theorems, of which theorem (3) is theorem I of the *Introduction.*

THEOREM 2. *If $A = a_1e_1 + \cdots + a_ne_n$ is any number of any given hypercomplex number system $e_1, e_2, \cdots, e_n$, either $A^m = 0$, for some positive integer $m \leqq n + 1$, or there is an idempotent polynomial $\mathfrak{f}(A)$ in A whose coefficients are rational functions in the domain R (1) of $a_1, a_2, \cdots, a_n$ and of the constants γ_{ijk} of multiplication.*

THEOREM 3. *In any hypercomplex number system there is an idempotent number, or every number of the system is nilpotent. Therefore, in a nilpotent system every number is nilpotent.* ‡

THEOREM 4. *Let R' denote the domain of rationality of any aggregate of scalars including the constants γ_{ijk} of multiplication of any given hypercomplex number system $e_1, e_2, \cdots, e_n$: either there is an idempotent number*

* See p. 513 above.

† See p. 513 above.

‡ PIERCE, loc. cit., p. 113.

belonging to $\mathfrak{B}(R'; e_1, e_2, \cdots e_n)$, *or the number system is nilpotent. In particular, in any real hypercomplex number system there is an idempotent number, or every number is nilpotent; and, therefore in a real nilpotent hypercomplex number system every number is nilpotent.*

By supposition, we have

$$\Omega(\lambda) \equiv \lambda^{\nu}(\lambda-\lambda_1)^{\nu_1}(\lambda-\lambda_2)^{\nu_2}\cdots(\lambda-\lambda_r)^{\nu_r}. \tag{34}$$

Let

$$W(\lambda) \equiv \lambda(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_r). \tag{35}$$

Then, *if any polynomial* $\sum_{q=1}^{q=p} c_q A^q$ *in* A *is nilpotent,* $\sum_{q=1}^{q=p} c_q \lambda^q$ *contains* $W(\lambda)$. Conversely, if $\sum_{q=1}^{q=p}\lambda^q$ contains $W(\lambda)$, $\sum_{q=1}^{q=p} c_q A^q$ is nilpotent, since then some power of $\sum_{q=1}^{q=p} c_q \lambda^q$ contains $\Omega(\lambda)$. By (30) and (32), we have

$$A^{\nu}[A-\lambda_1\mathfrak{f}(A)]^{\nu_1}[A-\lambda_2\mathfrak{f}(A)]^{\nu_2}\cdots[A-\lambda_r\mathfrak{f}(A)]^{\nu_r} = \Omega(A) = 0; \tag{36}$$

and, therefore, if m' is the greatest of the integers $\nu_1, \nu_2, \cdots, \nu_r$,

$$A^{\nu}\{[A-\lambda_1\mathfrak{f}(A)][A-\lambda_2\mathfrak{f}(A)]\cdots[A-\lambda_r\mathfrak{f}(A)]\}^{m'} = 0. \tag{37}$$

Further, if

$$A^{\nu}\{[A-\rho_1\mathfrak{f}(A)][A-\rho_2\mathfrak{f}(A)]\cdots[A-\rho_{r'}\mathfrak{f}(A)]\}^{p} = 0,$$

then $r' \geqq r$, and r of the scalars $\rho_1, \rho_2, \cdots, \rho_{r'}$ are equal, respectively, to $\lambda_1, \lambda_2, \cdots, \lambda_r$. For, from the last equation, it follows that

$$A^{\nu+r'} - \sum_{i=1}^{r'}\rho_i A^{\nu+r'-1} + \cdots + (-1)^{r'}\rho_1\rho_2\cdots\rho_{r'}A^{\nu}$$
$$= A^{\nu}[A-\rho_1\mathfrak{f}(A)][A-\rho_2\mathfrak{f}(A)]\cdots[A-\rho_{r'}\mathfrak{f}(A)],$$

by (30), is nilpotent; and, therefore,

$$\lambda^{\nu+r'} - \sum_{i=1}^{r'}\rho_i\lambda^{\nu+r'-1} + \cdots + (-1)^{r'}\rho_1\rho_2\cdots\rho_{r'}\lambda^{\nu}$$

contains $W(\lambda)$, which is impossible unless $r' \geqq r$ and the scalars $\rho_1, \rho_2, \cdots, \rho_{r'}$, comprise $\lambda_1, \lambda_2, \cdots, \lambda_r$.

Let

$$f(A) = \sum_{q=1}^{p} c_q A^q$$

be any polynomial in A such that

$$A^{\nu}f(A) = 0.$$

If $f(A)$ is not nilpotent, then by theorem (2), there is an idempotent polynomial

$$I = \sum_{q=1}^{p'}\gamma_q[f(A)]^q$$

in $f(A)$, and thus

$$A^{\nu}I = A^{\nu}\sum_{q=1}^{p'}\gamma_q[f(A)]^q = 0;$$

whence, since $\mathfrak{f}(A)$ is a polynomial in A^ν, it follows that

$$\mathfrak{f}(A)I = 0.$$

But, by (30), $\mathfrak{f}(A)$ is an idemfactor* of the νth power of any polynomial in A; and, therefore,

$$I = I^\nu = \mathfrak{f}(A)I^\nu = 0,$$

which is impossible. *Therefore, if*

$$A^\nu \sum_{q=1}^{p} c_q A^q = 0,$$

then

$$\sum_{q=1}^{p} c_q A^q$$

is nilpotent.† Whence, by (37), it follows that

$$\{[A-\lambda_1\mathfrak{f}(A)][A-\lambda_2\mathfrak{f}(A)]\cdots[A-\lambda_r\mathfrak{f}(A)]\}^{m'} = 0 \tag{38}$$

for some positive integer m'. Further, *if*

$$\{[(A-\rho_1\mathfrak{f}(A)][A-\rho_2\mathfrak{f}(A)]\cdots[A-\rho_{r'}\mathfrak{f}(A)]\}^p = 0,$$

then $r' \geqq r$ *and* r *of the scalars* $\rho_1, \rho_2, \cdots, \rho_{r'}$ *are equal, respectively, to* $\lambda_1, \lambda_2, \cdots, \lambda_r$. For, from the last equation, it follows that

$$A^\nu\{(A-\rho_1\mathfrak{f}(A)][A-\rho_2\mathfrak{f}(A)]\cdots[A-\rho_{r'}\mathfrak{f}(A)]\}^p = 0,$$

which, as proved above, is impossible unless $r' \geqq r$ and the scalars $\rho_1, \rho_2, \cdots, \rho_{r'}$ comprise $\lambda_1, \lambda_2, \cdots, \lambda_r$.‡

If $r \geqq 2$, the number system contains at least two independent idempotent

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*See p. 512 above.

†Similarly, if $A^{\nu'}f(A)$ is nilpotent, where $\nu' \leqq \nu$ and

$$f(A) = \sum_{q=1}^{p} e_q A^q,$$

then $A^\nu f(A)$ is nilpotent; and, therefore, $\mathfrak{f}(A)f(A)$, and thus $[f(A)]^\nu = [\mathfrak{f}(A)f(A)]^\nu$ is nilpotent, that is $f(A)$ is nilpotent. Let $\nu' > \nu$, and let $A^{\nu'}f(A)$ be nilpotent; then, by what has just be proved, $A^{\nu'-\nu}f(A)$ is nilpotent. Therefore, if $\nu' - \nu \leqq \nu$, $f(A)$ is nilpotent. On the other hand, if $\nu' - \nu > \nu$, then $A^{\nu'-2\nu}f(A)$ is nilpotent; and, proceeding thus, we may show ultimately that $f(A)$ is nilpotent. In particular, if $A^{\nu'}f(A) = 0$, $f(A)$ is nilpotent.

‡These results may be established as follows. For $1 \leqq i \leqq r$, the polynomial $\lambda - \lambda_i\mathfrak{f}(\lambda)$ contains both λ and $\lambda - \lambda_i$; whereas, $\lambda - \rho\mathfrak{f}(\lambda)$ contains $\lambda - \lambda_i$ only if $\rho = \lambda_i$. Therefore,

$$\prod_{i=1}^{r'}[\lambda - \rho_i\mathfrak{f}(\lambda)]$$

contains $W(\lambda)$ if $r' \geqq r$ and if the scalars $\rho_1, \rho_2, \cdots, \rho_{r'}$ comprise $\lambda_1, \lambda_2, \cdots, \lambda_r$, but not otherwise. Whence follow the theorems given above.

numbers. For then, by what has just been proved, neither $A - \lambda_r \mathfrak{f}(A)$ nor

$$\prod_{i=1}^{r-1} [A - \lambda_i \mathfrak{f}(A)]$$

is nilpotent; and, therefore, by theorem (2), there is an idempotent polynomial I in $A - \lambda_r \mathfrak{f}(A)$, and an idempotent polynomial J in

$$\prod_{i=1}^{r-1} (A - \lambda_i \mathfrak{f}(A)];$$

and, by (38)

$$IJ = I^{m'} J^{m'} = (IJ)^{m'} = 0,$$

which is impossible if $J = \rho I$.*

Let A belong to the domain $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, where R' contains $R(\gamma_{ijk})$; and let the factors of $\Omega(\lambda)$ rationally irreducible with respect to R' be

$$\lambda, \ \Omega_1(\lambda), \ \Omega_2(\lambda), \ \cdots, \ \Omega_s(\lambda),$$

so that

$$\Omega(\lambda) \equiv \lambda^{\nu} [\Omega_1(\lambda)]^{\nu_1'} [\Omega_2(\lambda)]^{\nu_2'} \cdots [\Omega_s(\lambda)]^{\nu_s'}. \tag{39}$$

For $1 \leqq i \leqq s$, the roots of $\Omega_i(\lambda) = 0$ are all distinct. Let

$$\begin{aligned} \Omega_i(\lambda) &\equiv \lambda^{r_i} + q_1^{(i)} \lambda^{r_i - 1} + \cdots + q_{r_i - 1}^{(i)} \lambda + q_{r_i}^{(i)} \\ &\equiv (\lambda - \lambda_1^{(i)})(\lambda - \lambda_2^{(i)}) \cdots (\lambda - \lambda_{r_i}^{(i)}) \end{aligned} \qquad (i = 1, 2, \cdots, s). \tag{40}$$

Then, for $1 \leqq i \leqq s$, $q_1^{(i)}, q_2^{(i)}, \cdots, q_{r_i}^{(i)}$ are rational in R'; and, therefore, since $\mathfrak{f}(A)$ belongs to $\mathfrak{R}(R'; e_1, e_2, e_n)$, so also does

$$\omega_i(A) \equiv [A - \lambda_1^{(i)} \mathfrak{f}(A)][A - \lambda_2^{(i)} \mathfrak{f}(A)] \cdots [A - \lambda_{r_i}^{(i)} \mathfrak{f}(A)] \qquad (i = 1, 2, \cdots, s). \tag{41}$$

By (38),

$$\begin{aligned} &[\omega_1(A)\omega_2(A) \cdots \omega_s(A)]^{m'} \\ &= \left(\prod_{i=1}^{s} \{[A - \lambda_1^{(i)} \mathfrak{f}(A)][A - \lambda_2^{(i)} \mathfrak{f}(A)][A - \lambda_{r_i}^{(i)} \mathfrak{f}(A)]\} \right)^{m'} = 0. \end{aligned} \tag{42}$$

Moreover, since no two of the equations $\Omega_1(\lambda) = 0$, $\Omega_2(\lambda) = 0$, etc., have a root in common, and zero is a root of neither of these equations,

$$\prod_{i=1}^{s'} \omega_i(A) = \prod_{i=1}^{s'} \{[A - \lambda_1^{(i)} \mathfrak{f}(A)][A - \lambda_2^{(i)} \mathfrak{f}(A)] \cdots [A - \lambda_{r_i}^{(i)} \mathfrak{f}(A)]\},$$

is not nilpotent if $s' < s$. Therefore if $s \geqq 2$, neither $\omega_s(A)$ nor

$$\prod_{i=1}^{s-1} \omega_i(A)$$

* For, if $IJ = 0$ and $J = \rho I$, then $\rho I = \rho I^2 = IJ = 0$, and thus $\rho = 0$; therefore, $J = 0$, which is contrary to supposition, since J is idempotent.

is nilpotent; and consequently, by theorem (2), there is an idempotent polynomial I in $\omega_s(A)$, and an idempotent polynomial J in

$$\prod_{i=1}^{s-1}\omega_i(A).$$

But, by (42)

$$IJ = I^{m'}J^{m'} = (IJ)^{m'} = 0,$$

which is impossible if $J = \rho I$. Since $\omega_i(A)$ belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, so also do I and J. Therefore, if $s \geqq 2$, the number system contains at least two independent idempotent numbers belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$.

Let now the number system $e_1, e_2, \cdots, e_n$ contain but one idempotent number I. Then by what has just been proved, if $A^m \neq 0$,

$$\Omega(\lambda) \equiv \lambda^\nu(\lambda - \alpha)^{n_1},$$

where $\alpha \neq 0$. Further, $\mathfrak{f}(A) = I$; and, therefore, by (38),

$$A - \alpha I = A - \alpha\mathfrak{f}(A)$$

is nilpotent. Therefore,

$$A = \alpha I + N$$

where $N = A - \alpha I = A - \alpha\mathfrak{f}(A)$ is nilpotent. If, moreover,

$$A = \rho I + N',$$

when N' is nilpotent, then, by theorem (1),

$$\rho SI = S(\rho I + N') = SA = S(\alpha I + N) = \alpha SI;$$

and, therefore, since $SI \neq 0$, $\rho = \alpha$ and, thus, $N = N'$. If A is nilpotent, and

$$A = \rho I + N$$

where N is nilpotent, by theorem (1),

$$\rho SI = S(\rho I + N) = SA = 0;$$

and, therefore, $\rho = 0$. Let R' include $R(\gamma_{ijk})$. At least one of the units $e_1, e_2, \cdots, e_n$ is not nilpotent; otherwise, by theorem (1), the system is nilpotent. Therefore, there is at least one non-nilpotent number A belonging to $\mathfrak{R}[R(\gamma_{ijk}); e_1, e_2, \cdots, e_n]$; and, therefore, since $I = \mathfrak{f}(A)$, I belongs to $\mathfrak{R}[R(\gamma_{ijk}); e_1, e_2, \cdots, e_n]$. If A belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, and $A^m \neq 0$, then α is rational in R'; and, therefore, $N = A - \alpha I$ belongs to $\mathfrak{R}(R', e_1, e_2, \cdots, e_n)$. Whence follows

THEOREM 5. *Let the number system $e_1, e_2, \cdots, e_n$ contain one and only idempotent number I; and let R' denote the domain of rationality of any aggregate of scalars including the constants γ_{ijk} of multiplication of the sys-*

tem. Then I belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$. Moreover, any number A of the system is separable in one and only one way into the sum $\alpha I + N$ of a scalar multiple of I and a nilpotent number N, which is commutative with A. If A is not nilpotent, and $\Omega(A)$ is the fundamental equation of A, then

$$\Omega(\lambda) \equiv \lambda^{\nu}(\lambda - \alpha)^{\nu_1}.$$

Finally, if A belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, N belongs to this domain, and α is rational in R'.

Let

$$e_i = \alpha_i I + N_i \qquad (i = 1, 2, \cdots, n),$$

where $N_i (i = 1, 2, \cdots, n)$ is nilpotent. Then, by the preceding theorem, I and $N_i (1 \leqq i \leqq n)$ belong to $\mathfrak{R}[R(\gamma_{ijk}); e_1, e_2, \cdots, e_n]$, and $\alpha_i (1 \leqq i \leqq n)$ is rational in $R(\gamma_{ijk})$. The idempotent number I and $n-1$ of the units are independent. Thus let I and $e_1, e_2, \cdots, e_{n-1}$ be independent. Then I and $N_1, N_2, \cdots, N_{n-1}$ are independent; and, therefore, we may substitue

$$e_n' = I, \qquad e_i' = N_i = e_i - \alpha_i I \qquad (i = 1, 2, \cdots, n-1),$$

for the original units. Since each of the new units belongs to

$$\mathfrak{R}[R(\gamma_{ijk}); e_1, e_2, \cdots, e_n],$$

the transformation is rational in $R(\gamma_{ijk})$. Wherefore,

THEOREM 6. *If the number system $e_1, e_2, \cdots, e_n$ contains but one idempotent number we may substitute, by a transformation rational in $R(\gamma_{ijk})$, new units $e_1', e_2', \cdots, e_n'$ such that*

$$e_n'^2 = e_n', \qquad e_i'^{m_i} = 0 * \qquad (i = 1, 2, \cdots, n-1).$$

Let the number system $e_1, e_2, \cdots, e_n$ contain but one idempotent number I belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, where $\mathfrak{R}'$ includes $R(\gamma_{ijk})$; and let A be any non-nilpotent number of the system belonging to this domain. Then, by what precedes,

$$\Omega(\lambda) \equiv \lambda^{\nu}[\Omega_1(\lambda)]^{\nu_1},$$

where

$$\Omega_1(\lambda) \equiv \lambda^{r_1} + q_1^{(1)}\lambda^{r_1-1} + \cdots + q_{r_1-1}^{(1)}\lambda + q_{r_1}^{(1)}$$

is rationally irreducible in R'. Since A belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, so also does $\mathfrak{f}(A)$; and, therefore, $\mathfrak{f}(A) = I$. By (29)

$$A^{\nu}[A^{r_1} + q_1^{(1)}A^{r_1-1} + \cdots + q_{r_1-1}^{(1)}A + q_{r_1}^{(1)}\mathfrak{f}(A)]^{\nu_1} = \Omega(A) = 0;$$

and therefore, by the theorem given on p. 527,

* This theorem includes the first part of theorem IV of the *Introduction*.

$$(A^{r_1} + q_1^{(1)} A^{r_1-1} + \cdots + q_{r_1-1}^{(1)} A + q_{r_1}^{(1)} I)^{m'}$$
$$= [A^{r_1} + q_1^{(1)} A^{r_1-1} + \cdots + q_{r_1-1}^{(1)} A + q_{r_1}^{(1)} \mathfrak{f}(A)]^{m'} = 0,$$

for some positive integer m'. Whence follows

THEOREM 7. *Let the hypercomplex number system $e_1, e_2, \cdots, e_n$ contain but one idempotent number I belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, where R' denotes the domain of rationality of any aggregate of scalars including the constants γ_{ijk} of multiplication of the number system. Then, if A is any non-nilpotent number of the system belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, and $\Omega(A) = 0$ is the fundamental equation of A,*

$$\Omega(\lambda) \equiv \lambda^{\rho} [\Omega_1(\lambda)]^{r_1},$$

where $\Omega_1(\lambda)$ is rationally irreducible in R'. Moreover, if

$$\Omega_1(\lambda) \equiv \lambda^{r_1} + q_1^{(r)} \lambda^{r_1-1} + \cdots + q_{r_1-1}^{(1)} \lambda + q_{r_1}^{(1)},$$

then

$$(A^{r_1} + q_1^{(1)} A^{r_1-1} + \cdots + q_{r_1-1}^{(1)} A + q_{r_1}^{(1)} I)^{m'} = 0,$$

for some positive integer m'.

Let now $e_1, e_2, \cdots e_n$ be the units of any given number system containing at least one nilpotent number; and let A be any nilpotent number, other than zero, of the system, — thus let $A^{m-1} \neq 0$, $A^m = 0$. Then, by theorem (1), $SA^p = 0$ for any positive integer p; and, therefore,

$$SA^{m-1} = SA^{m-2} = \cdots = SA = 0.$$

Conversely, let m be the order of the fundamental equation of A, and let A satisfy the above conditions. Then A is nilpotent. For, otherwise, by theorem (2), there is an idempotent polynomial $\mathfrak{f}(A)$ in A; but, by the theorem given p. 523, it follows from the above equations that $SA^p = 0$ for any positive integer p, and thus $S\mathfrak{f}(A) = 0$, which is impossible since $\mathfrak{f}(A)$ is idempotent. Therefore,

THEOREM 8. *If m is the order of the fundamental equation of any number $A \neq 0$ of a given hypercomplex number system, and*

$$SA = SA^2 = \cdots = SA^{m-1} = 0,$$

a fortiori, if $SA^p = 0$ for any positive integer p, then A is nilpotent. Conversely, if A is nilpotent $SA^p = 0$ for any positive integer p.

§ 3. *Classification of the units of an idempotent hypercomplex number system.**

Let R' denote the domain of rationality of any aggregate of scalars comprising the constants γ_{ijk} of multiplication of the idempotent number system

* See p. 512 above.

$e_1, e_2, \cdots, e_n$. By theorem (4) there is at least one idempotent number belonging to the domain $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$. Let

$$I = \sum_{i=1}^{n} a_i^{(0)} e_i$$

be any idempotent number belonging to this domain; and, for $i = 1, 2, \cdots, n$, let

$$\begin{aligned} e_i^{(1)} &= Ie_iI, \\ e_i^{(2)} &= Ie_i - Ie_iI, \\ e_i^{(3)} &= e_iI - Ie_iI, \\ e_i^{(4)} &= e_i - Ie_i - e_iI + Ie_iI. \end{aligned} \tag{43}$$

We have, for $i = 1, 2, \cdots, n$,

$$\begin{aligned} Ie_i^{(1)} &= e_i^{(1)} = e_i^{(1)}I, \\ Ie_i^{(2)} &= e_i^{(2)}, \qquad e_i^{(2)}I = 0, \\ Ie_i^{(3)} &= 0, \qquad e_i^{(3)}I = e_i^{(3)}, \\ Ie_i^{(4)} &= 0 = e_i^{(4)}I; \end{aligned} \tag{44}$$

and thus, for $1 \leqq i \leqq n$, the numbers $e_i^{(1)}, e_i^{(2)}, e_i^{(3)}, e_i^{(4)}$ are, respectively, in the first, second, third, and fourth of PEIRCE's groups (mentioned in theorem II of the *Introduction*) with respect to the idempotent number I as the basis.* Further, any number linear in the $e^{(k)}$'s $(k = 1, 2, 3, 4)$ belongs to the kth groups with respect to I.† Since

$$e_i = e_i^{(1)} + e_i^{(2)} + e_i^{(3)} + e_i^{(4)} \qquad (i = 1, 2, \cdots, n), \tag{45}$$

every number of the system can be expressed linearly in the $4n$ numbers of (43).

Let n_1 of the numbers $e_i^{(1)}$ of the first group with respect to I, as $e_1^{(1)}, e_2^{(1)}, \cdots, e_{n_1}^{(1)}$, be independent; n_2 of the numbers $e_i^{(2)}$ of the second group, as $e_1^{(2)}, e_2^{(2)}, \cdots, e_{n_2}^{(2)}$; n_3 of the numbers $e_i^{(3)}$ of the third group, as $e_1^{(3)}, e_2^{(3)}, \cdots, e_{n_3}^{(3)}$; and, finally, let n_4 of the numbers $e_i^{(4)}$ of the fourth group, as $e_1^{(4)}, e_2^{(4)}, \cdots, e_{n_4}^{(4)}$, be independent. Then since each of the numbers $e_i^{(k)}$, for $k = 1, 2, 3, 4$ and $i = 1, 2, \cdots, n$ (and, therefore, any n independent numbers of the system) can be expressed in terms of the above $n_1 + n_2 + n_3 + n_4$ hypercomplex numbers, and the latter are independent,‡ we have $n_1 + n_2 + n_3 + n_4 = n$. We may now show that any number of the system belonging to the kth group with respect to I $(1 \leqq k \leqq 4)$ is linear in $e_1^{(k)}, e_2^{(k)}, \cdots, e_{n_k}^{(k)}$. Whence it follows that I is linear in

* See p. 509 above.

† See p. 512 above.

‡ There can be no linear relation between numbers belonging to different groups with respect to I. Cf. the third footnote on p. 518.

$e_1^{(1)}, e_2^{(1)}, \cdots, e_{n_1}^{(1)}$. We may, therefore, substitute I for one of these numbers. It may be that either n_2, n_3, or n_4 is zero; but $n_1 \geqq 1$. For n_1 is the number of independent hypercomplex numbers in the aggregate Ie_iI for $i = 1, 2, \cdots, n$; and these n numbers cannot all be zero, that is $n_1 \neq 0$, otherwise,

$$I = I^3 = I\sum_{i=1}^{n} a_i^{(0)} e_i \cdot I = \sum_{i=1}^{n} a_i^{(0)} Ie_iI = 0.$$

Since both I and the units e_1, e_2, etc., belong to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, so also do the numbers $e_i^{(k)}$. Therefore, the transformation from the original system of units to the new system of units,

$$\begin{aligned}
e_i' &= e_i^{(1)} && (i = 1, 2, \cdots, n_1), \\
e_{n_1+i}' &= e_i^{(2)} && (i = 1, 2, \cdots, n_2), \\
e_{n_1+n_2+i}' &= e_i^{(3)} && (i = 1, 2, \cdots, n_3), \\
e_{n_1+n_2+n_3+i}' &= e_i^{(4)} && (i = 1, 2, \cdots, n_4),
\end{aligned} \tag{46}$$

is rational in R'; and, therefore, the domain $\mathfrak{R}(R'; e_1', e_2', \cdots, e_n')$ is identical with the domain $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$.* The same is true if we substitute I for one of the units of the first group.

Let us now suppose that the first group with respect to I contains a second idempotent number J belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots e_n)$. We may then by a transformation rational in R' substitute for the second system of units, regular† with respect to I as the basis, a third system of units regular with respect to J as the basis, each of the units of the third system belonging to $\mathfrak{R}(R'; e_1', e_2', \cdots, e_n')$, and therefore, to $\mathfrak{R}(R'; e_1, e_2 \cdots e_n)$. The number of units of the third system in the first group with respect to J is the number of independent hypercomplex numbers in the aggregate $Je_i^{(k)}J$ for $k = 1, 2, 3, 4$ and $i = 1, 2, \cdots, n_k$. Since $IJ = J = JI$, we have

$$Je_i^{(k)}J = JI \cdot e_i^{(k)} \cdot IJ = J \cdot Ie_i^{(k)}I \cdot J = 0$$

for $k > 1$, $i = 1, 2, \cdots, n_k$; and, therefore, the number of units of the third system in the first group with respect to J cannot exceed n_1. But, if

$$I - J = c_1 e_1^{(1)} + c_2 e_2^{(1)} + \cdots + c_{n_1} e_{n_1}^{(1)},$$

then

$$\sum_{i=1}^{n_1} c_i Je_i^{(1)}J = J \cdot \sum_{i=1}^{n_1} c_i e_i^{(1)} \cdot J = J(I - J)J = J - J = 0;$$

and, therefore, the number of units of the third system in the first group with respect to J is less than n_1. Thus ultimately, by repeated transformations each

*See p. 514 above.

†See p. 512 above.

rational in R' we can feel free the first group of any idempotent number belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$ other than the basis. In the system of units which we shall thus ultimately obtain, we may take the basis as one of the units of the first group. We have, therefore, the following theorem.

THEOREM (9). *Let R' denote the domain of rationality of any aggregate of scalars including the constants γ_{ijk} of multiplication of the idempotent number system $e_1, e_2, \cdots, e_n$. Then, there is at least one idempotent number I belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$; and we may, by a transformation rational in R', introduce new units $e_1', e_2', \cdots, e_n'$ regular with respect to I as the basis, each of the new units belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, and this domain being identical with $\mathfrak{R}(R'; e_1', e_2', \cdots, e_n')$. We may take I as one of the units $e_1', e_2', \cdots, e_n'$. If the first group with respect to I of the system of units $e_1', e_2', \cdots, e_n'$ contains a second idempotent number I' belonging to $\mathfrak{R}(R'; e_1', e_2', \cdots, e_n')$, we may by a second transformation rational in R' introduce a third system of units $e_1'', e_2'', \cdots, e_n''$ (of which we may take I' to be one) regular with respect to I' as the basis; each of these units will then belong to $\mathfrak{R}(R'; e_1', e_2', \cdots, e_n')$, this domain being identical with $\mathfrak{R}(R'; e_1'', e_2'', \cdots, e_n'')$; and the number of units in the first group with respect to I' of the system $e_1'', e_2'', \cdots, e_n''$ will be at least one less than the number of units of the second system in the first group with respect to I. Therefore, we may introduce, by a transformation rational in R', new units $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_n$ regular with respect to an idempotent number $\bar{I}$ belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, and such that the first group with respect to $\bar{I}$ contains no second idempotent number; the domain $\mathfrak{R}(R'; \bar{e}_1, \bar{e}_2, \cdots, \bar{e}_n)$ will then be identical with $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$. We may take I as one of the units of the final system.*

After the substitution, by a transformation rational in R', of a new system of units, $e_i^{(k)}$ ($k = 1, 2, 3, 4$ and $i = 1, 2, \cdots, n_k$) regular with respect to an idempotent number I belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, let us suppose that the fourth group with respect to I contains an idempotent number J belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$. By a transformation rational in R' we may introduce a third system of units regular with respect to J. The units of the third system in the first groups with respect to J are, then, the independent hypercomplex numbers in the aggregate $Je_i^{(k)}J$ for $k = 1, 2, 3, 4$ and $i = 1, 2, \cdots, n_k$. But, since $IJ = JI = 0$, we have

$$I \cdot Je_i^{(k)}J = 0 = Je_i^{(k)}J \cdot I \quad (k = 1, 2, 3, 4; i = 1, 2, \cdots, n_k); \tag{47}$$

and, therefore, the units of the third system in the first group with respect to J are in the fourth group with respect to I. The units of the third system in the second group with respect to J are the independent hypercomplex numbers in the aggregate $Je_i^{(k)} - Je_i^{(k)}J$ for $k = 1, 2, 3, 4$ and $i = 1, 2, \cdots, n_k$. But, by (44),

$$Je_i^{(1)} - Je_i^{(1)}J = J\cdot Ie_i^{(1)} - J\cdot Ie_i^{(1)}\cdot J = 0 \qquad (i=1, 2, \cdots, n_1), \tag{48}$$

$$Je_i^{(2)} - Je_i^{(2)}J = J\cdot Ie_i^{(2)}\cdot J - J\cdot Ie_i^{(2)}\cdot J = 0 \qquad (i=1, 2, \cdots, n_2); \tag{49}$$

and also

$$Je_i^{(3)}J = J\cdot e_i^{(3)}I\cdot J = 0 \qquad (i=1, 2, \cdots, n_3),$$

whence follows

$$(Je_i^{(3)} - Je_i^{(3)}J)I = Je_i^{(3)}\cdot I = Je_i^{(3)} = Je_i^{(3)} - Je_i^{(3)}J \qquad (i=1, 2, \cdots, n_3). \tag{50}$$

Moreover, by (44), we have

$$(Je_i^{(4)} - Je_i^{(4)}J)I = Je_i^{(4)}I = 0 \qquad (i=1, 2, \cdots, n_4). \tag{51}$$

Finally,

$$I(Je_i^{(k)} - Je_i^{(k)}J) = 0 \qquad (k=3, 4;\ i=1, 2, 3, 4). \tag{52}$$

Therefore, by (48) to (52), the units of the third system in the second group with respect to J are each, severally, in one of PEIRCE's four groups with respect to I,—namely, in either the third or fourth group with respect to I. Similarly, we may show that each of the units of the third system in the third group with respect to J is in either the second or fourth group with respect to I; and that each of the units of the third system in the fourth group with respect to J is in either the first, second, third, or fourth group with respect to I. Since $IJ = JI = 0$, we may take both I and J as units of the third system. We have, therefore, the following theorem:

THEOREM. 10. *Let R' denote the domain of rationality of any aggregate of scalars including the constants of multiplication of the idempotent number system $e_1, e_2, \cdots, e_n$; and let the units of the system be regular with respect to an idempotent number I belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$. Then, if the fourth group with respect to I contains an idempotent number belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, we may, by a transformation rational in R', introduce new units $e_1', e_2', \cdots, e_n'$ which shall be regular both with respect to I and with respect to J. Of this system we may take both I and J as units.*

When R' includes every scalar real and imaginary, this theorem reduces to the theorem on which HAWKES bases his method for regularizing a number system with respect to each of the idempotent numbers chosen as units.*

§ 4. *Nilpotent hypercomplex number system.*

Let $e_1, e_2, \cdots, e_n$ be the units of a nilpotent hypercomplex number system. Then, by theorem (3), *every number of the system is nilpotent.* Further, by theorem (8), *if A is any number of the system, $SA = 0$*; in particular,

$$Se_1 = Se_2 = \cdots = Se_n = 0. \tag{53}$$

* HAWKES, loc. cit., p. 317.

Conversely, if the scalar of each of any n independent numbers of any number system in n units is zero, the system is nilpotent*. Whence, or directly from theorem (3), it follows that *every subsystem of a nilpotent system is nilpotent.*

In consequence of a well-known theorem of SCHEFFERS', relating to non-quaternion number systems,† it follows that, in a nilpotent numbert system, new units $e_1', e_2', \cdots, e_n'$ can be so selected that

$$e_i' e_j' = \sum_{k=h}^{n} \gamma_{ijk}' e_k' \qquad (i, j = 1, 2, \cdots n;\ h > i;\ h > j), \tag{54}$$

that is, such that the constants γ_{ijk}' of multiplication of the new units satisfy the conditions

$$\gamma_{ijk}' = 0 \ddagger \quad (i, j = 1, 2, \cdots, n;\ k \leqq i;\ k \leqq j). \tag{55}$$

The transformation by which this choice of units is effected can be taken rational in $\mathfrak{R}[R(\gamma_{ijk});\ e_1, e_2, \cdots, e_n]$. For this is true for $n = 1$, in which case $e_1^2 = 0$. Again for $n = 2$, if there is a number $A = a_1 e_1 + a_2 e_2$ belonging to $\mathfrak{R}[R(\gamma_{ijk});\ e_1, e, \cdots, e_n]$ whose square is not zero, we may put

$$e_1' = A, \qquad e_2' = A^2,$$

when we have

$$e_1'^2 = e_2', \qquad e_1' e_2' = e_2' e_1' = e_2'^2 = 0.\S$$

If there is no number belonging to $\mathfrak{R}[R(\gamma_{ijk});\ e_1, e, \cdots, e_n]$ whose square is not zero, we have $e_1 e_2 + e_2 e_1 = 0$;‖ and, if $e_1 e_2 \neq 0$, we may put

$$e_1' = e_1, \qquad e_2' = e_1 e_2,$$

when we have

$$e_1'^2 = 0, \qquad e_1' e_2' = e_2' e_1' = e_2'^2 = 0;\P$$

whereas, if $e_1 e_2 = 0$ (in which case $e_2 e_1 = 0$), the identical transformation,

$$e_1' = e_1, \qquad e_2' = e_2$$

satisfies the requirements. Therefore, for $n = 1$ and $n = 2$, new units can be

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* Theorem (1).

† *Encyclopädie der Mathematischen Wissenschaften*, vol. 1, p. 181.

‡ HAWKES, loc. cit., p. 323.

§ The non-vanishing powers of a nilpotent number are independent, PEIRCE, loc. cit., p. 114. Therefore, $A^3 = 0$.

‖ For in the case supposed $e_1^2 = 0$, $e_2^2 = 0$, $(e_1 + e_2)^2 = 0$; whence follows $e_1 e_2 + e_2 e_1 = 0$.

¶ We have $e_1'^2 = 0$, $e_2'^2 = 0$, and $e_1' e_2' = e_1^2 e_2 = 0$ by supposition. If

$$e_2' e_1' = e_1 e_2 \cdot e_1 = c_1 e_1' + c_2 e_2' = c_1 e_1 + c_2 e_1 e_2,$$

then since

$$0 = e_1 e_2 \cdot e_1^2 = c_1 e_1^2 + c_2 e_1 e_2 \cdot e_1 = c_2 e_1 e_2 \cdot e_1,$$

either $e_2' e_1' = e_1 e_2 \cdot e_1 = 0$, or we have $c_2 = 0$, in which case $e_1 e_2 \cdot e_1 = c_1 e_1$, and, therefore,

$$0 = e_1 e_2 \cdot e_1 e_2 = c_1 e_1 e_2,$$

that is $c_1 = 0$.

chosen to accord with (55) by a transformation rational in $R(\gamma_{ijk})$. Finally, we may show that, if new units can be selected to accord with (55) by a transformation rational in $R(\gamma_{ijk})$ for every nilpotent system in less than n units, the same is true for the given system in n units.

In demonstrating this theorem, I shall assume that, for every nilpotent number system in less than n units, new units can be obtained by a transformation rational in $R(\gamma_{ijk})$ to accord with (55): first, I show by the aid of theorems (i) to (iii) below, irrespective of the rationality of the transformation for a system with less than n units, and without recourse to the theory of groups, that in any given nilpotent system with n units there are numbers (one or more) nilfacient and nilfaciend to every number of the system; second, that certain of these numbers belong to $\mathfrak{R}[R(\gamma_{ijk}); e_1, e_2, \cdots, e_n]$, follows from simple considerations; third, from the existence of such numbers belonging to $\mathfrak{R}[R(\gamma_{ijk}); e_1, e_2, \cdots, e_n]$, the lemma given at the end of this section, and our assumption, it follows that the theorem assumed for less than n units is true also of the given nilpotent system with n units.

Let now $A = \sum_{i=1}^{i=n} a_i e_i \neq 0$ be any number of the nilpotent system $e_1, e_2, \cdots, e_n$. Then, since A is nilpotent, it is nilfacient to at least one number of the system. Let A be nilfacient to just m independent numbers, as $A_1, A_2, \cdots, A_m$, of the system. Then, in the first place, A is nilfacient to any number linear in $A_1, A_2, \cdots, A_m$. Secondly, these numbers form a subsystem by themselves. For, if $m = n$, $e_1, e_2, \cdots, e_n$ can be expressed linearly in $A_1, A_2, \cdots, A_n$; on the other hand, if $m < n$, and, for $1 \leqq i \leqq m$, $1 \leqq j \leqq m$, we have

$$A_i A_j = \sum_{h=1}^{m} c_h A_h + B,$$

where B is independent of $A_1, A_2, \cdots, A_m$, then

$$0 = AA_i \cdot A_j = A \cdot A_i A_j = \sum_{h=1}^{m} c_h AA_h + AB = AB,$$

which is contrary to supposition.* The system $A_1, A_2, \cdots, A_m$ being identical with $e_1, e_2, \cdots, e_n$, or a subsystem of the latter, is nilpotent. We may now interchange the order of multiplication and the terms nilfacient and nilfaciend, when we have

THEOREM (i). If A is nilfacient (nilfaciend) to m hypercomplex numbers, $A_1, A_2, \cdots, A_m$, it is nilfacient (nilfaciend) to any number linear in $A_1, A_2, \cdots, A_m$.

THEOREM (ii). Any number $A \neq 0$ of a nilpotent system $e_1, e_2, \cdots, e_n$ is

*CARTAN, Annales de la Faculté des Sciences de Toulouse, vol. 12 (1898).

nilfacient (nilfaciend) to just m independent hypercomplex numbers, where $1 \leqq m \leqq n$, and these numbers form a nilpotent system by themselves. *

Let $A \neq 0$ be nilfacient or nilfaciend to just $m < n$ independent numbers of the system $e_1, e_2, \cdots, e_n$, as $A_1, A_2, \cdots, A_m$. Then by theorem (ii) $A_1, A_2, \cdots, A_m$ form by themselves a nilpotent subsystem; and, since $m < n$, by our assumption, this subsystem contains m independent numbers $A_1', A_2', \cdots, A_m'$ such that

$$A_i' A_j' = \sum_{k=h}^{m} \alpha_{ijk} A_k' \qquad (i, j = 1, 2, \cdots, m;\ h > i;\ h > j). \tag{56}$$

I shall now show that there is a number B independent of $A_1', A_2', \cdots, A_m'$ such that

$$BA_i' = b_{i1} A_1' + b_{i2} A_2' + \cdots + b_{im} A_m' \qquad (i = 1, 2, \cdots, m). \tag{57}$$

First, if there is any number B_m such that $B = B_m A_m'$ is independent of the A''s, then since, by (56),

$$BA_i' = B_m A_m' \cdot A_i' = B_m A_m' A_i' = 0 \qquad (i = 1, 2, \cdots, m),$$

there is a number B independent of the A''s satisfying equations (57). We may, therefore, assume that the product of any number of the system $e_1, e_2, \cdots, e_n$ multiplied by A_m' as post factor is linear in the A''s. Second, if there is any number B_{m-1} such that $B = B_{m-1} A_{m-1}'$ is independent of the A''s, then since, by (56),

$$BA_i' = B_{m-1} A_{m-1}' \cdot A_i' = B_{m-1} \cdot A_{m-1}' A_i' = \alpha_{m-1\,im} B_{m-1} A_m' \ (i = 1, 2, \cdots, m-1),$$

and both BA_m' and $B_{m-1} A_m'$ are linear in the A''s by supposition, it follows that there is a number B independent of the A''s satisfying equations (57). We may, therefore, assume that the product of any number of the system $e_1, e_2, \cdots, e_n$ multiplied by A_i' $(i = m-1, m)$ as post factor is linear in the A''s. Proceeding thus, we find either a number B independent of the A''s satisfying equations (57); or that the product of any number of the system by A_i' $(i = 2, 3, \cdots, m)$ as post factor is linear in the A''s. We may, therefore, assume that the last is true. Finally, if there is any number B_1 such that $B = B_1 A_1'$ is independent of the A''s, then since by (56),

$$BA_i' = B_1 A_1' \cdot A_i' = B_1 \cdot A_1' A_i' = \sum_{k=h}^{m} \alpha_{1ik} B_1 A_k' \qquad (i = 1, 2, \cdots, m;\ 2 \leqq h > i),$$

and since $B_1 A_k'$ for $k \geqq 2$ is linear in the A''s by supposition, there is a number B independent of the A''s satisfying equations (57). If there is no such number B_1, then the product of any number of the system by A_i' $(i = 1, 2, \cdots, m)$ as post factor is linear in the A''s; and, therefore since $m < n$, there is a num-

* CARTAN. See preceding note.

ber B independent of the A''s satisfying equations (57). Similarly, we may show that there is a number C independent of the A''s such that

$$A'_i C = c_{i1} A'_1 + c_{i2} A'_2 + \cdots + c_{im} A'_m \qquad (i=1, 2, \cdots, m). \tag{58}$$

Since B is nilpotent, some power of B is linear in the A''s. Similarly, some power of C is linear in the A''s. Let B^p and C^q, respectively, be the highest powers of B and C independent of the A''s; then

$$B^{p+1} = \beta_1 A'_1 + \beta_2 A'_2 + \cdots + \beta_m A'_m, \tag{59}$$

$$C^{q+1} = \gamma_1 A'_1 + \gamma_2 A'_2 + \cdots + \gamma_m A'_m. \tag{60}$$

From (57) and (58), we derive

$$B^p A'_i = b^{(p)}_{i1} A'_1 + b^{(p)}_{i2} A'_2 + \cdots + b^{(p)}_{im} A'_m \qquad (i=1, 2, \cdots, m), \tag{61}$$

$$A'_i C^q = c^{(q)}_{i1} A'_1 + c^{(q)}_{i2} A'_2 + \cdots + c^{(q)}_{im} A'_m \qquad (i=1, 2, \cdots, m), \tag{62}$$

Therefore, by (56), (59), and (61),

$$\begin{aligned} A'_m B^p \cdot B = A'_m \cdot B^{p+1} = 0, \\ A'_m B^p \cdot A'_i = A'_m \cdot B^p A'_i = 0 \end{aligned} \qquad (i=1, 2, \cdots, m); \tag{63}$$

and, by (56), (60), and (62),

$$\begin{aligned} C \cdot C^q A'_m = C^{q+1} \cdot A'_m = 0, \\ A'_i \cdot C^q A'_m = A'_i C^q \cdot A'_m = 0 \end{aligned} \qquad (i=1, 2, \cdots, m). \tag{64}$$

Let us now assume, first, that $A^{(1)} \equiv A'_m B^p \neq 0$.* This number is then, by (63), nilfacient to at least $m+1$ independent numbers $A'_1, A'_2, \cdots, A'_m, B$; and is, moreover, nilfaciend to $A'_1, A'_2, \cdots, A'_m$. If $A^{(1)} = A'_m B^p$ is nilfaciend to some number independent of the A''s, it is, then, also nilfaciend to at least $m+1$ independent numbers. On the other hand, if $A^{(1)} = A'_m B^p$ is nilfaciend to no number independent of the A''s, then, since C^q is independent of the A''s, $A^{(2)} \equiv C^q A'_m B^p \neq 0$. But in this case, the number $A^{(2)} = C^q A'_m B^p$ is nilfaciend to at least $m+1$ independent numbers, since by (64),

$$\begin{aligned} C \cdot C^q A'_m B^p = C^{q+1} A'_m \cdot B^p = 0, \\ A'_i \cdot C^q A'_m B^p = A'_i C^q A'_m \cdot B^p = 0 \end{aligned} \qquad (i=1, 2, \cdot\ , m); \tag{}$$

and is, moreover, nilfacient to at least $m+1$ independent numbers, namely, $A'_1, A'_2, \cdots, A'_m, B$. Second, let $A'_m B^p = 0$, then since B^p is independent of the A''s, $A^{(3)} \equiv A'_m$ is nilfacient to at least $m+1$ independent numbers. If

* Since A'_m may be nilfacient (or nilfaciend) to numbers of the system other than $A'_1, A'_2, \cdots, A'_m$, it may happen $A'_m B^p = 0$ (or $C^q A'_m = 0$).

now $A^{(3)} = A'_m$ is nilfaciend to some number independent of the A''s, it is, then, also nilfaciend to at least $m+1$ independent numbers. On the other hand, if $A^{(3)} = A'_m$ is nilfaciend to no number independent of the A''s, then, since C^q is independent of the A''s, $A^{(4)} \equiv C^q A'_m \neq 0$. But in this case, the number $A^{(4)} = C^q A'_m$, which is nilfacient to at least $m+1$ independent numbers (namely, those to which A'_m is nilfacient), is also nilfaciend to at least $m+1$ independent numbers, since by (64),

$$C \cdot C^q A'_m = C^{q+1} \cdot A'_m = 0,$$
$$A'_i \cdot C^q A'_m = A_i C^q \cdot A'_m = 0 \qquad (i = 1, 2, \cdots, m).$$

Whence follows

THEOREM (iii). If the number $A \neq 0$ of a nilpotent system is nilfacient (nilfaciend) to just $m < n$ independent numbers, there is a number A' nilfacient to at least $m+1$ independent numbers and nilfaciend to at least $m+1$ independent numbers.

Let now $A \neq 0$ be any given number of the system, then, by theorem (ii), A is nilfacient to at least one number and nilfaciend to at least one number of the system. Therefore, if A is not both nilfacient and nilfaciend to n independent numbers, it is either nilfacient or nilfaciend to just $m < n$ independent numbers, in which case, by the last theorem, there is a number A' nilfacient to at least $m+1$ independent numbers and nilfaciend to at least $m+1$ independent numbers. Therefore, if A' is not nilfacient and nilfaciend to n independent numbers, it is either nilfacient or nilfaciend to just $m' < n$ independent numbers. But, since $m' \geqq m+1$, proceeding thus, we must ultimately obtain a number both nilfacient and nilfaciend to n independent numbers, and, therefore, nilfacient and nilfaciend to every number of the system.

The conditions necessary and sufficient that $A = \sum_{i=1}^{i=n} a_i e_i$ shall be nilfacient and nilfaciend to every number of the system are, by theorem (i),

$$\sum_{i=1}^{n}\sum_{k=1}^{n} a_i \gamma_{ijk} e_k = \sum_{i=1}^{n} a_i e_i e_j = A e_j = 0 \qquad (j = 1, 2, \cdots, n),$$

that is,

$$a_1\gamma_{ijk} + a_2\gamma_{2jk} + \cdots + a_n\gamma_{njk} = 0 \qquad (j, k = 1, 2, \cdots, n),$$

and

$$\sum_{i=1}^{n}\sum_{k=1}^{n} a_i \gamma_{1ik} e_k = \sum_{i=1}^{n} a_i e_j e_i = e_j A = 0 \qquad (j = 1, 2, \cdots, n),$$

that is,

$$a_1\gamma_{j1k} + a_2\gamma_{j2k} + \cdots + a_n\gamma_{jnk} = 0 \qquad (j, k = 1, 2, \cdots, n).$$

Since there is at least one such number A, there is at least one such number A belonging to $\mathfrak{R}[R(\gamma_{ijk}); e_1, e_2, \cdots e_n]$. We may, therefore, by a transformation rational in $R(\gamma_{ijk})$, introduce new units $e'_1, e'_2, \cdots, e'_n$ such that

$$e'_n e'_i = 0 = e'_i e'_n \qquad (i = 1, 2, \cdots, n),$$

the constants γ'_{ijk} of multiplication of the new units being rational in $R(\gamma_{ijk})$. By the lemma given below, there is, then, a nilpotent number system $\bar{e}_1, \bar{e}_2, \cdots \bar{e}_{n-1}$ such that

$$\bar{e}_i \bar{e}_j = \sum_{k=1}^{n-1} \gamma'_{ijk} \bar{e}_k \qquad (i, j = 1, 2, \cdots, n-1).$$

By our assumption, we may introduce into this number system, by a transformation

$$\bar{e}'_i = \sum_{k=1}^{n-1} \tau_{ik} \bar{e}_k \qquad (i = 1, 2, \cdots, n-1)$$

rational in $R(\gamma'_{ijk})$, and, therefore, rational in $R(\gamma_{ijk})$, new units $\bar{e}'_1, \bar{e}'_2, \cdots, \bar{e}'_{n-1}$ to accord with (55), that is, such that

$$\bar{e}'_i \bar{e}'_j = \sum_{k=h}^{n-1} \gamma''_{ijk} \bar{e}'_k \quad (i, j = 1, 2, \cdots, n-1;\ h > i;\ h > j).$$

When, by the lemma, if

$$e''_i = \sum_{k=1}^{n-1} \tau_{ik} e'_k \qquad (i = 1, 2, \cdots, n-1),$$

$$e''_n = e'_n,$$

we have

$$e''_i e''_j = \sum_{k=h}^{n-1} \gamma''_{ijk} e''_k + \gamma''_{ijn} e''_n \quad (i, j = 1, 2, \cdots, n;\ h > i;\ h > j),$$

$$e''_n e''_i = 0 = e''_i e''_n \qquad (i = 1, 2, \cdots, n).$$

This transformation is also rational in $R(\gamma'_{ijk})$ and, therefore, in $R(\gamma_{ijk})$; and, thus, by two successive transformations, each rational in $R(\gamma_{ijk})$, we may obtain a system of units whose multiplication table accords with (55).*

THEOREM 11. *In any nilpotent number system $e_1, e_2, \cdots, e_n$ we may substitute, by a transformation rational in $R(\gamma_{ijk})$, new units $e'_1, e'_2, \cdots, e'_n$ such that*

$$e'_i e'_j = \sum_{k=h}^{n} \gamma_{ijk} e'_k \quad (i, j = 1, 2, \cdots, n;\ h > i;\ h > j).$$

LEMMA. Let $e_1, e_2, \cdots, e_n$ be any number system for which

$$e_n e_i = 0 = e_i e_n \qquad (i = 1, 2, \cdots, n).$$

Then, denoting by γ_{ijk} the constants of multiplication of this system, there is a number system $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{n-1}$ in $n-1$ units such that

$$\bar{e}_i \bar{e}_j = \sum_{k=1}^{n-1} \gamma_{ijk} \bar{e}_k \qquad (i, j = 1, 2, \cdots, n-1),$$

* Since the constants of multiplication of the system $e''_1, e''_2, \cdots, e''_n$ are rational functions of the coefficients of the transformation for the domain $R(\gamma'_{ijk})$, they are rational in the domain $R(\gamma_{ijk})$.

which is nilpotent if the system $e_1, e_2, \cdots, e_n$ is nilpotent. Moreover, if we introduce new units $e_1', e_2', \cdots, e_n'$ and $\bar{e}_1', \bar{e}_2', \bar{e}_{n-1}'$ into the respective systems $e_1, e_2, \cdots, e_n$ and $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{n-1}$ by transformations

$$e_i' = \sum_{k=1}^{n-1} \tau_{ik} e_k, \qquad e_n' = e_n \qquad (i=1, 2, \cdots, n-1),$$

and

$$\bar{e}_i' = \sum_{k=1}^{n-1} \tau_{ik} \bar{e}_k \qquad (i=1, 2, \cdots, n-1);$$

and, if

$$e_i' e_j' = \sum_{k=1}^{n} \gamma'_{ijk} e_k' \qquad (i, j=1, 2, \cdots, n),$$

$$\bar{e}_i' \bar{e}_j' = \sum_{1}^{n-1} \bar{\gamma}_{ijk} \bar{e}_k' \qquad (i, j=1, 2, \cdots, n-1),$$

then

$$\gamma'_{ijk} = \bar{\gamma}_{ijk} \qquad (i, j, k=1, 2, \cdots, n-1).$$

For let $e_1, e_2, \cdots, e_n$ be the units of any hypercomplex number system such that

$$\gamma_{nij} = 0 = \gamma_{inj} \qquad (i, j=1, 2, \cdots, n),$$

that is, such that $e_n e_i = 0 = e_i e_n$ for $i = 1, 2, \cdots, n$. Then

$$\sum_{h=1}^{n-1} \gamma_{ihl}\gamma_{jkh} = \sum_{h=1}^{n-1} \gamma_{ihl}\gamma_{jkh} + \gamma_{inl}\gamma_{jkn} = \sum_{h=1}^{n} \gamma_{ihl}\gamma_{jkh} = \sum_{h=1}^{n} \gamma_{ijh}\gamma_{hkl}$$

$$= \sum_{h=1}^{n-1} \gamma_{ijh}\gamma_{hkl} + \gamma_{ijn}\gamma_{nkl} = \sum_{h=1}^{n-1} \gamma_{ijh}\gamma_{hkl} \qquad (i, j, k, l=1, 2, \cdots, n-1),$$

which is the condition necessary and sufficient that a number system $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{n-1}$ shall exist such that

$$\bar{e}_i \bar{e}_j = \sum_{h=1}^{n-1} \gamma_{ijh} \bar{e}_h \text{*} \qquad (i, j=1, 2, \cdots, n-1).$$

Let

$$(a_1 e_1 + a_2 e_2 + \cdots + a_n e_n)(b_1 e_1 + b_2 e_2 + \cdots + b_n e_n) = (c_1 e_1 + c_2 e_2 + \cdots + c_n e_n),$$

$$(a_1 \bar{e}_1 + \cdots + a_{n-1} \bar{e}_{n-1})(b_1 \bar{e}_1 + \cdots + b_{n-1} \bar{e}_{n-1}) = (\bar{c}_1 \bar{e}_1 + \cdots + \bar{c}_{n-1} \bar{e}_{n-1}).$$

Then, since

$$\bar{c}_k = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a_i b_j \gamma_{ijk} \qquad (k=1, 2, \cdots, n-1),$$

and

$$c_k = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \gamma_{ijk} = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a_i b_j \gamma_{ijk} \quad (k=1, 2, \cdots, n-1),$$

we have

$$\bar{c}_k = c_k \qquad (k=1, 2, \cdots, n-1).$$

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* See p. 515 above.

Let now $\bar{e}'_1, \bar{e}'_2, \cdots, \bar{e}'_{n-1}$ and $e'_1, e'_2, \cdots, e'_n$ be new systems of units of the respective number systems $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{n-1}$ and $e_1, e_2, \cdots, e_n$, obtained, respectively, by the linear transformations $\overline{T}$ and T of non-zero determinant defined, respectively, as follows,

$$\bar{e}'_i = \tau_{i1}\bar{e}_1 + \tau_{i2}\bar{e}_2 + \cdots + \tau_{in-1}\bar{e}_{n-1} \qquad (i=1, 2, \cdots, n-1),$$

and

$$e'_i = \tau_{i1}e_1 + \tau_{i2}e_2 + \cdots + \tau_{in-1}e_{n-1}, \qquad e'_n = e_n \qquad (i=1, 2, \cdots, n-1),$$

and let

$$\bar{e}'_i\bar{e}'_j = \sum_{h=1}^{n-1}\bar{\gamma}'_{ijh}\bar{e}'_h = \sum_{h=1}^{n-1}\sum_{k=1}^{n-1}\bar{\gamma}'_{ijh}\tau_{hk}\bar{e}_k \qquad (i, j=1, 2, \cdots, n-1),$$

$$e'_i e'_j = \sum_{h=1}^{n}\gamma'_{ijh}e'_h = \sum_{h=1}^{n-1}\sum_{k=1}^{n-1}\gamma'_{ijh}\tau_{hk}e_k + \gamma'_{ijn}e_n \qquad (i, j=1, 2, \cdots, n-1).$$

Then, by what has just been proved,

$$\sum_{h=1}^{n-1}\bar{\gamma}'_{ijh}\tau_{hk} = \sum_{h=1}^{n-1}\gamma'_{ijh}\tau_{hk} \qquad (i, j, k=1, 2, \cdots, n-1)$$

and therefore

$$\bar{\gamma}'_{ijk} = \gamma'_{ijk} \qquad (i, j=1, 2, n-1);$$

for, otherwise, the determinant of $\overline{T}$ is zero, which is contrary to supposition.

The number systems $e_1, e_2, \cdots, e_n$ and $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{n-1}$ have the same number of independent idempotent numbers. For, if

$$(65) \qquad (a_1^{(k)}e_1 + \cdots + a_n^{(k)}e_n)^2 = a_1^{(k)}e_1 + \cdots + a_n^{(k)}e_n \qquad (k=1, 2, \cdots, m),$$

then

$$(66) \qquad a_n^{(k)} = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}a_i^{(k)}a_j^{(k)}\gamma_{ijn} \qquad (k=1, 2, \cdots, m),$$

and

$$(67) \qquad (a_1^{(k)}\bar{e}_1 + \cdots + a_{n-1}^{(k)}\bar{e}_{n-1})^2 = a_1^{(k)}\bar{e}_1 + \cdots + a_{n-1}^{(k)}\bar{e}_{n-1} \qquad (k=1, 2, \cdots, m).$$

Conversely, from (66) and (67) we obtain (65). Moreover, if

$$\begin{Vmatrix} a_1^{(1)} & a_2^{(1)} & \cdots & a_n^{(1)} \\ a_1^{(2)} & a_2^{(2)} & \cdots & a_n^{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ a_1^{(m)} & a_2^{(m)} & \cdots & a_n^{(m)} \end{Vmatrix} = 0,$$

then

$$\begin{Vmatrix} a_1^{(1)} & a_2^{(1)} & \cdots & a_{n-1}^{(1)} \\ a_1^{(2)} & a_2^{(2)} & \cdots & a_{n-1}^{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ a_1^{(m)} & a_2^{(m)} & \cdots & a_{n-1}^{(m)} \end{Vmatrix} = 0;$$

and, conversely, by (66). Whence it follows that, if $e_1, e_2, \cdots, e_n$ is nilpotent, so also is $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{n-1}$; and conversely.*

§5. *Hypercomplex number system with but one idempotent number.*

Let $e_1, e_2, \cdots, e_n$ be the units of a hypercomplex number system with one and only one idempotent number

$$I = \sum_{i=1}^{n} a_i^{(0)} e_i,$$

in particular, $e_1, e_2, \cdots, e_n$ may be the units of the first group with respect to I of a number system in $n' \geqq n$ units; and let R' denote the domain of rationality of any aggregate of scalars including the constants γ_{ijk} of multiplication of the number system. Then, by theorem (5), I belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$. Let

$$A = \sum_{i=1}^{n} a_i e_i$$

be any number of the system, and let

$$C = IAI - IA;$$

then

$$C^2 = 0, \qquad IC = C, \qquad CI = 0.$$

If $C \neq 0$, there is a second idempotent number in the system, namely, $I + \rho C$ for any scalar $\rho \neq 0$. For, we have

$$(I + \rho C)^2 = I + \rho(IC + CI) + \rho^2 C^2 = I + \rho C;$$

and, if $I + \rho C = \sigma I$, then

$$(1 - \sigma) I = [(1 - \sigma) I + \rho C] I = 0;$$

thus, $\sigma = 1$ and, therefore, $\rho = 0$, which is contrary to supposition. Therefore,

$$IAI - IA = 0.$$

Similarly, we may show that

$$IAI - AI = 0.$$

Whence it follows that

$$IA = IAI = AI; \tag{68}$$

that is, every number of the system is commutative with I.

* Or, otherwise, as follows. We have

$$\sum_{j=1}^{n} \gamma_{ij1} = \sum_{j=1}^{n-1} \gamma_{ij1}, \qquad \sum_{j=1}^{n} \gamma_{nj1} = 0 \qquad (i = 1, 2, \ldots, n-1).$$

Therefore, if $Se_i = 0$ $(i = 1, 2, \cdots n)$, then $S\bar{e}_i = 0$ $(i = 1, 2, \cdots n-1)$; and, conversely. See theorem (11).

By theorem (5) any number A of the system is separable in one and only one way into the sum of a scalar multiple of I and a nilpotent number. Let

$$A = \alpha I + N \tag{69}$$

where N is nilpotent. Since N is commutative with I, it is also commutative with A.*

By theorem (1),

$$SA = S(\alpha I + N) = \alpha SI, \tag{70}$$

where $SI \neq 0$. Therefore, if $SA = 0$, $\alpha = 0$, and $A = N$ is nilpotent; conversely, if A is nilpotent, $SA = 0$ by theorem (8).†

Since $IN = NI$, IN is nilpotent. Therefore,

$$SIA = S \cdot I(\alpha I + N) = S(\alpha I + IN) = \alpha SI = SA. \tag{71}$$

Let now A be nilpotent. Then

$$SIA = SA = 0,$$

and IA is nilpotent; and, therefore, there is a smallest positive integer p for which $(IA)^p = 0$. Let B be any second number of the system. If $p = 1$, then

$$SAB = S(I \cdot AB) = S(IA \cdot B) = 0.$$

On the other hand, if $p > 1$, and

$$AB = \rho I + N',$$

where $N'^{m'} = 0$, then

$$IA \cdot B = I \cdot AB = \rho I + IN',$$

and, therefore,

$$\rho(IA)^{p-1} + (IA)^{p-1}N' = \rho(IA)^{p-1}I + (IA)^{p-1}IN' = (IA)^{p-1}(\rho I + IN') = (IA)^p B = 0,$$

*See theorem (5). Further, if A is not nilpotent, $N = A - \alpha I = A - \alpha f(A)$; whereas, on the other hand, if A is nilpotent $N = A$, see p. 529. Therefore, in either case, N is a polynomial in A.

†Again, since A is separable in but one way into the sum $\alpha I + N$ of a scalar multiple α of I and a nilpotent number N, A is nilpotent if and only if $\alpha = 0$, and, therefore, if and only if $SA = 0$.

Let $\Omega(\lambda) = 0$ be the fundamental equation of the number A of any given idempotent number system; and let m_0 be the greatest number of distinct roots, other than zero, of $\Omega(\lambda) = 0$ for any number A of the system. Then the condition necessary and sufficient that an arbitrarily given number A of the system shall be nilpotent is $SA^p = 0$ for $p = 1, 2, \cdots, m_0$. If the given idempotent system contains but one idempotent number I belonging to $\mathfrak{N}(R'; e_1, e_2, \cdots, e_n)$, where R' contains $R(\gamma_{ijk})$ and A belongs to $\mathfrak{N}(R'; e_1, e_2, \cdots, e_n)$, $\Omega(\lambda) \equiv \lambda^\nu[\Omega_1(\lambda)]^{\nu_1}$, where $\Omega_1(\lambda)$ is irreducible with respect to R'; and, therefore, if m_0 is the highest degree of $\Omega_1(\lambda)$ for any number A of the system, the condition necessary and sufficient that A shall be nilpotent is $SA^p = 0$ for $p = 1, 2, \cdots, m_0$.

that is,

$$(IA)^{p-1}N' = -\rho(IA)^{p-1};$$

whence follows

$$(-\rho)^{m'}(IA)^{p-1} = (-\rho)^{m'-1}(IA)^{p-1}N' = \cdots = -\rho(IA)^{p-1}N'^{m-1}$$
$$= (IA)^{p-1}N'^{m'} = 0;$$

and, thus, $\rho = 0$. Therefore, also for $p > 1$

$$SAB = \rho SI = 0.$$

Whence it follows that AB is nilpotent if A is nilpotent; and, since $SBA = SAB$, BA is also nilpotent if A is nilpotent.

If $SA = 0$, $SB = 0$, then

$$S(A + B) = SA + SB = 0;$$

and, therefore, the sum of two nilpotent numbers is nilpotent.

Let A and B be any two numbers of the system. Since

$$S(SI \cdot A - SA \cdot I) = SA \cdot SI - SA \cdot SI = 0,$$

and, therefore, $SI \cdot A - SA \cdot I$ is nilpotent, by what precedes,

$$SI \cdot SAB - SA \cdot SB = SI \cdot SAB - SA \cdot SIB = S(SI \cdot A - SA \cdot I)B = 0,$$

that is,

$$SI \cdot SAB = SA \cdot SB. \tag{72}$$

Let

$$N^m = (A - \alpha I)^m = 0.$$

Then, if $SA \neq 0$, and thus $\alpha \neq 0$,

$$\left(I + \frac{N}{\alpha}\right)\left[I - \frac{N}{\alpha} + \left(\frac{N}{\alpha}\right)^2 - \cdots + (-1)^{m-1}\left(\frac{N}{\alpha}\right)^{m-1}\right]I$$
$$= I + (-1)^m\left(\frac{N}{\alpha}\right)^m I = I;$$

and, therefore, if

$$A^{(1)} \equiv \frac{1}{\alpha}\left[I + \left(I - \frac{A}{\alpha}\right) + \left(I - \frac{A}{\alpha}\right)^2 + \cdots + \left(I - \frac{A}{\alpha}\right)^{m-1}\right]I \tag{73}$$

we have

$$A^{(1)}A = I = AA^{(1)}. \tag{74}$$

If A belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, where R' contains $R(\gamma_{ijk})$, then N belongs to this domain and α is rational in R' by theorem (5); and, therefore, in this case, $A^{(1)}$ belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$.

By theorem (6) the number system contains $n - 1$ independent nilpotent

numbers which belong to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, where R' contains $R(\gamma_{ijk})$. The system cannot contain n independent nilpotent numbers $A_1, A_2, \cdots, A_n$. For, we should then have

$$I = c_1 A_1 + c_2 A_3 + \cdots + c_n A_n;$$

and, therefore

$$SI = c_1 SA_1 + c_2 SA_2 + \cdots + c_n SA_n = 0,$$

which is impossible.* We may take as units of our system $e'_n = I$ and any $n-1$ independent nilpotent numbers $e'_1, e'_2, \cdots, e'_{n-1}$ belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$. The numbers $e'_1, e'_2, \cdots, e'_{n-1}$ form a nilpotent system by themselves. For, let

$$e'_i e'_j = \gamma'_{ij1} e'_1 + \gamma'_{ij2} e'_2 + \cdots + \gamma'_{ijn-1} e'_{n-1} + \gamma'_{ijn} e'_n \quad (i, j = 1, 2, \cdots, n-1).$$

Then, by (72),

$$0 = \frac{1}{SI} Se'_i \cdot Se'_j = Se'_i e'_j = \sum_{k=1}^{n} \gamma'_{ijk} Se'_k = \gamma_{ijn} SI \qquad (i, j = 1, 2, \cdots, n-1);$$

and, therefore,

$$\gamma_{ijn} = 0 \qquad (i, j = 1, 2, \cdots, n).$$

Moreover, since $Se'_i = 0$ for $i = 1, 2, \cdots, n-1$, the system $e'_1, e'_2, \cdots, e'_{n-1}$ is nilpotent.

By theorem (11) we may substitute for the units of this nilpotent subsystem $n-1$ units $e''_1, e''_2, \cdots, e''_{n-1}$ such that

$$e''_i e''_j = \sum_{k=h}^{n-1} \gamma''_{ijk} e''_k \quad (i, j = 1, 2, \cdots, n-1; h > i; h > j).$$

We have now the following theorem, the latter part of which includes theorem IV of the *Introduction.*

THEOREM 12. *Let the hypercomplex number system $e_1, e_2, \cdots, e_n$ contain one and only one idempotent number I; and let A be any number of the system. Then $IA = AI$; and the condition necessary and sufficient that A shall be nilpotent is $SA = 0$. If B is any second number of the system, we have*

$$SI \cdot SAB = SA \cdot SB.$$

Furthermore, the sum of two nilpotent numbers is nilpotent, and any product is nilpotent of which one factor is nilpotent.† *Finally, if R' denotes the*

*This suffices to demonstrate the theorem proved in the next paragraph. For let $A_1, A_2, \cdots, A_{n-1}$ be any $n-1$ independent nilpotent numbers of the system. Then, since $A_i A_j$ $(1 \leqq i \leqq n-1, 1 \leqq j \leqq n-1)$ is nilpotent, it is linear in $A_1, A_2, \cdots, A_{n-1}$; otherwise, there are n independent nilpotent numbers in the system.

† For any given hypercomplex number system containing but one idempotent number belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, where R' contains $R(\gamma_{ijk})$, it is also true, for numbers belonging to this hypercomplex domain, that both the sum of two nilpotent numbers, and any product of which one factor is nilpotent, are nilpotent.

domain of rationality of any aggregate of scalars including the constants γ_{ijk} *of multiplication of the system, there are systems of just* $n-1$ *independent nilpotent numbers belonging to* $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$*; and by a transformation rational in* R' *we may substitute new units* $e'_n = I$*, and* $n-1$ *nilpotent numbers* $e'_1, e'_2, \cdots, e'_{n-1}$ *such that*

$$e'_i e'_j = \sum_{k=h}^{n-1} \gamma'_{ijk} e'_k \qquad (i, j = 1, 2, \cdots, n-1;\ h > i;\ h > j),$$

$$e'_n e'_i = e'_i = e'_i e'_n \qquad (i = 1, 2, \cdots, n),$$

the nilpotent numbers $e'_1, e'_2, \cdots, e'_{n-1}$ *thus forming a nilpotent subsystem.*

It follows as a corollary of this theorem that, in a number system regular with respect to an idempotent number I and containing no second idempotent number in the first group, no product of a number in the second group as facient and a number in the third group as faciend can contain a term involving I unless the fourth group contains one or more indempotent numbers.

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NOTES AND ERRATA: VOLUME 1, 3, 4, 5

VOLUME 1

F. R. MOULTON: *On a class of particular solutions* ....

P. 28, ll. 6 and 5 up. The numerical specifications are correct; the notice to the contrary (*Notes and errata*, vol. 3, p. 499) is in error. —F. R. M.

VOLUME 3

E. V. HUNTINGTON: *A complete set of postulates* ....

P. 267, l. 16 up. *For* one and only one *read* at least one.

E. H. MOORE: *A definition of abstract groups.*

P. 490, ll. 5–13. The independence of the postulates of

$$(M'') = (1, 2, 3'', 3''_l, 3''_r, 4''_l)$$

is not established, for the reasoning of the paragraph, ll. 5–13, p. 490, is in error, viz., (l. 5) from $(3_l, 3_r)$ does not follow $(3'', 3''_l, 3''_r)$, and (l. 12) the example for (3_r) in (M) does not suffice for $(3''_r)$ in (M'').

Now, in fact, in (M'') the postulate $(3''_r)$ is redundant, and we have the interesting definition,

$$(M''): (1, 2, 3'', 3''_l, 4''_l).$$

In M'' the postulates are mutually independent, and the same thing remains true when we add to the postulates of of M'' the postulate that the multiplication or composition of elements is commutative. For proofs of the statements here made I refer to a note to appear in volume 6 of the Transactions.—E. H. M.

VOLUME 4

L. E. DICKSON: *Definitions of a field by independent postulates.*

P. 19. As pointed out by Dr. E. V. HUNTINGTON, system $\Sigma_{8'}$ forms a field, 8′ being satisfied by $u = -2$. I find that the following system

○	0	1
0	0	1
1	1	0

□	0	1
0	0	0
1	0	0

may be employed.—L. E. D.

E. KASNER: *The generalized Beltrami problem* ···.

P. 149, l. 4 up. *For* point of transformation *read* point transformation.

E. J. WILCZYNSKI: *On a certain congruence* ···.

P. 190, l. 33. It is also assumed that $u_{11} = 0$, i. e. that S' is developable.

P. 191, l. 4. *For* $4q_{22}$ *read* $4q'_{22}$.

P. 194. The proof here given for the case $\theta_4 = 0$ is incorrect, owing to the fact that $v_{12} = 0$. It is still true that the surface S belongs to a linear complex; it has in fact a straight line directrix. Moreover all surfaces S' of the congruence Γ in this case have the property in question, instead of merely a double infinity.—E. J. W.

E. KASNER: *On the point-line as element* ···.

Pp. 213–233. Since the publication of this essay, I have learned of the memoir by Professor E. VENERONI, *Sopra i connessi bilineari fra punti e rette nello spazio ordinario*, published in the Memorie di Torino, ser II^a, vol. 4 (1902), pp. 115–159, in which many of my results are anticipated. The two treatments of the bilinear connex are quite independent. VENERONI's includes an elaborate discussion of the various special cases, while mine is devoted entirely to the general case. Among the results, concerning this general case, which do not appear in the Italian memoir may be mentioned: the group of the connex and its relation to known groups, the general representation of the various invariants and covariants arising, the normal form with respect to cogredient transformation, and the determination of the connex from its fundamental configuration. — E. K.

P. 214, l. 10. *For the second exponent* m *read* n.

P. 231, l. 18. " *the subscripts* 4 " κ.

L. E. DICKSON: *On the subgroups of order* ···.

P. 371, l. 8. *For* canonican *read* canonical.

P. 375. A better notation for the set $\Sigma_{k,\gamma}$ is $\Sigma_{k,\theta}$, where $\theta \equiv \gamma k - c^2$, so that both subscripts are now invariant.

P. 376, l. 20. *For* p^{n-1} *read* $(p^n - 1)/(p - 1)$. The same correction should be made five times in the theorem on p. 377.

H. A. MERRILL: *On solutions of differential equations.*

P. 432, l. 8 up. *For* . *read* , in which A, B and C are independent of λ.

S. EPSTEEN, *Semireducible hypercomplex number systems.*

Pp. 437–444. I desire to point out the relation of the systems which are semireducible of the first kind to the imprimitive (nichtursprüngliche) system of MOLIEN in Mathematische Annalen, vol. 41. This can be done best by means of the following table (cf. the table of vol. 3, p. 442).—S. E.

Conditions on Number System.	Name of System.	Group.
$A1, A2, C1, C2$ (Transactions, vol. 3, pp. 440, 442).	Semi-reducible of the first kind.	G is reducible, G_{11} is the group of E_1, G_{22} is not necessarily the group of E_2.
$A2, C1, C2$ (Mathematische Annalen, vol. 41, pp. 9–23).	Imprimitive.	G is reducible, G_{11} is the group of the *accompanying system* (not necessarily E_1) and G_{22} is not necessarily the group of E_2.

VOLUME 5

L. E. DICKSON: *The subgroups of order a power of* 2 ⋯.

P. 2, l. 12. In $\Omega_{2,5}$ replace 13 by 13^2.

L. E. DICKSON: *Determination of all the subgroups* ⋯.

P. 166, l. 13. *For* H_{212} *read* H_{216}.

E. W. BROWN: *On the smaller perturbations* ⋯.

P. 284, l. 7 up. *For* $\sin V'' + V' - 2h'')$ *read* $\sin (V'' + V' - 2h'')$.

" l. 4 up. " $a' a'' (V'' + V' - 2h'')$ " $a' a'' \cos (V'' + V' - 2h'')$.

P. 285, l. 2. " D^{-n} " D_0^{-n}.

E. V. HUNTINGTON: *Sets of independent postulates for the algebra of logic.*

P. 292, 2d footnote. Further essays of LEIBNIZ's, more numerous and extensive than those included in GERHARDT's edition, will be found in a volume of *Opuscules inédits de Leibnis*, published by L. COUTURAT in 1903.—E. V. H.

J. B. SHAW: *Algebras defined by finite groups.*

P. 326, footnote ‡. It seems to have been CAYLEY who first suggested the correspondence in question; cf. *On the theory of groups as depending on the symbolic equation* $\theta^n = 1$, Philosophical Magazine, vol. 7 (1854), pp. 40–47, or *Collected Works*, vol. 2, paper number 125, p. 129.—J. B. S.